

Regular characters of $GL_n(O)$ and Weil representations over finite fields

Koichi Takase

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1 Introduction

Let F be a non-Archimedean local field, O its integer ring, \mathfrak{p} the maximal ideal of O which is generated by ϖ . The residue class field O/\mathfrak{p} is denoted by \mathbb{F} which is a finite field of q elements. Let us denote by O_l the residue class ring O/\mathfrak{p}^l for a positive integer l so that $O_1 = \mathbb{F}$. Fix a continuous unitary character τ of the additive group F such that

$$\{x \in F \mid \tau(xy) = 1 \text{ for all } y \in O\} = O.$$

Define a unitary character $\hat{\tau}$ of the additive group \mathbb{F} by $\hat{\tau}(\bar{x}) = \tau(\varpi^{-1}x)$.

The irreducible unitary representations of $GL_n(O)$ may play an important role in the harmonic analysis of $GL_n(F)$, because the group $GL_n(O)$ is a maximal compact subgroup of the locally compact group $GL_n(F)$. Since each irreducible unitary representation of $GL_n(O)$ factors through a finite group $G_r = GL_n(O_r)$ for some positive integer r , the classification of the irreducible unitary representations of $GL_n(O)$ is reduced to determine the irreducible complex representations of the finite group G_r . The classification for the case $r = 1$ is given by J.A.Green [3]. Although the complete classification of the irreducible representations of G_r , for the case $r > 1$, is a *wild problem* [12, p.4417], a remarkable partial classification is given by G.Hill [4, 5, 6, 7]. Let us recall his approach.

For any integer $0 < i < r$, let us denote by K_i the kernel of the canonical surjective group homomorphism $G_r \rightarrow G_i$. Then the mapping $1_n + \varpi^i X \pmod{\mathfrak{p}^r} \mapsto X \pmod{\mathfrak{p}}$ gives a surjective group homomorphism of K_i onto the additive group $M_n(\mathbb{F})$. For any \mathbb{F} -vector subspace U of $M_n(\mathbb{F})$, let us denote by $K_i(U)$ the inverse image of U under the group homomorphism, which is a normal subgroup of K_i .

Let l be the smallest integer such that $r/2 \leq l$ and put $l + l' = r$. Then K_l is isomorphic to the additive group $M_n(O_{l'})$ via the group isomorphism $1_n + \varpi^l X \pmod{\mathfrak{p}^r} \mapsto X \pmod{\mathfrak{p}^{l'}}$. For any $\beta \in M_n(O)$, let us denote by ψ_β the character of K_l defined by

$$\psi_\beta(k) = \tau\left(\varpi^{-l'} \text{tr}(X\beta)\right)$$

for $k = 1_n + \varpi^l X \pmod{\mathfrak{p}^r} \in K_l$. Then $\beta \pmod{\mathfrak{p}^{l'}} \mapsto \psi_\beta$ gives an isomorphism of the additive group $M_n(O_{l'})$ to the character group of K_l . For each ψ_β , let

us denote by $\text{Irr}(G_r \mid \psi_\beta)$ the equivalence classes of the irreducible complex representation π of G_r such that

$$\langle \psi_\beta, \pi \rangle_{K_l} = \dim_{\mathbb{C}} \text{Hom}_{K_l}(\psi_\beta, \pi) > 0.$$

Note that $GL_n(O)$ -conjugation of β gives the same $\text{Irr}(G_r \mid \psi_\beta)$. Note also that, for any $\gamma \in GL_n(O)$, we have $\psi_{\gamma\beta\gamma^{-1}}(k) = \psi_\beta(\gamma^{-1}k\gamma)$. So let us denote by $G_r(\psi_\beta)$ the subgroup of $\gamma \pmod{\mathfrak{p}^r} \in G_r$ such that $\gamma\beta\gamma^{-1} \equiv \beta \pmod{\mathfrak{p}^r}$. Note that K_l is contained in $G_r(\psi_\beta)$. Then Clifford theory says that $\text{Irr}(G_r \mid \psi_\beta)$ is determined by the set $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$ of the equivalence class of irreducible complex representation σ of $G_r(\psi_\beta)$ such that $\langle \psi_\beta, \sigma \rangle_{K_l} > 0$. More precisely,

$$\sigma \mapsto \text{Ind}_{G_r(\psi_\beta)}^{G_r} \sigma$$

gives a bijection of $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$ onto $\text{Irr}(G_r \mid \psi_\beta)$. So Hill's approach is to determine the set $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$ according to the specific property of $\bar{\beta} = \beta \pmod{\mathfrak{p}} \in M_n(\mathbb{F})$.

Suppose that $\bar{\beta} \in M_n(\mathbb{F})$ is regular, that is, the characteristic polynomial $\chi_{\bar{\beta}}(t)$ of $\bar{\beta}$ is equal to its minimal polynomial. Then we have ([7, Cor. 3.9])

$$G_r(\psi_\beta) = (O_r[\beta_r])^\times K_{l'}$$

where $\beta_r = \beta \pmod{\mathfrak{p}^r} \in M_n(O_r)$ (the residue class of $X \in M_n(O)$ modulo a power \mathfrak{p}^i with $i > 1$ is denoted by $X_i \in M_n(O_i)$ whenever there is no risk of confusion). For the sake of simplicity let us denote $\mathcal{C} = (O_r[\beta_r])^\times$. If r is even, then $l' = l$ and all the elements of $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$ are one-dimensional which are determined by the extensions to \mathcal{C} of $\psi_\beta|_{\mathcal{C} \cap K_l}$.

Theorem 4.6 in his paper [7], Hill treats the case where $r = 2l - 1$ is odd and the regular $\bar{\beta} \in M_n(\mathbb{F})$ is split, that is the eigenvalues of $\bar{\beta}$ are contained in \mathbb{F} . Unfortunately the proof of the theorem works only for a semi-simple $\bar{\beta}$ as shown by Proposition 2.1.1 of this paper.

Shintani [11] and Gérardin [2] treat the case where $r = 2l - 1$ is odd and the characteristic polynomial of $\bar{\beta}$ is irreducible over \mathbb{F} (the cuspidal case). However their method is rather complicated because it depends on the parity of n and treats two cases separately.

In this paper we will assume that $\bar{\beta} \in M_n(\mathbb{F})$ is regular and $r = 2l - 1$ is odd and will give a method to describe the set $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$ in a uniform manner under a hypothesis that a certain Schur multiplier associated with a symplectic space over finite field is trivial (Theorem 3.2.2). This hypothesis is valid if the characteristic polynomial of $\bar{\beta}$ is separable (Theorem 5.2.1) so that the cuspidal case is included in this case. In the case of split $\bar{\beta}$, we will give evidences good enough to call the hypothesis a conjecture (Theorem 5.3.1).

Our basic idea is that the group K_{l-1} is very close to the Heisenberg group associated with a symplectic space over a finite field. So we can construct an irreducible representation of K_{l-1} via the Schrödinger representation. Then the theory of Weil representation [13] enable us to extend the irreducible representation of K_{l-1} to a representation of $\mathcal{C}K_{l-1}$, but there appears an obstruction to the extension (section 5). This obstruction is described by certain Schur multiplier with which we will mainly concern in this paper. The section 4 is devoted to the study of it.

The characteristic of the finite field \mathbb{F} is arbitrary in sections 2 and 3. It is assumed to be odd in sections 4 and 5.

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2 Remarks on a result of Hill

2.1 In this section, we will assume that $r = 2l - 1$ is odd and that $\bar{\beta} \in M_n(\mathbb{F})$ is regular and split. So, after taking suitable $GL_n(O)$ -conjugate of β , we can assume that $\bar{\beta}$ is in a Jordan canonical form

$$\bar{\beta} = \begin{bmatrix} J_{n_1}(a_1) & & \\ & \ddots & \\ & & J_{n_f}(a_f) \end{bmatrix}, \quad (1)$$

where $a_1, \dots, a_f \in \mathbb{F}$ are the distinct eigenvalues of $\bar{\beta}$ and

$$J_m(a) = \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & \\ & & & a \end{bmatrix} \in M_m(\mathbb{F})$$

is a Jordan block. Let W be the \mathbb{F} -vector subspace of the upper triangular matrices in $M_n(\mathbb{F})$. Let us denote by $X(\psi_\beta)$ the set of the one-dimensional representation ψ of $K_{l-1}(W)$ such that $\psi|_{K_l} = \psi_\beta$. Then we have

$$\sharp X(\psi_\beta) = (K_{l-1}(W) : K_l) = |W| = q^{n(n+1)/2},$$

because we have $K_l \subset K_{l-1}(W)$ and $[K_{l-1}(W), K_{l-1}(W)] \subset \text{Ker}(\psi_\beta)$. An action of $k \in K_{l-1}$ on $\psi \in X(\psi_\beta)$ is defined by $\psi^k(x) = \psi(kxk^{-1})$. Then [7] shows, in the proof of Theorem 4.6, that the isotropy subgroup of any $\psi \in X(\psi_\beta)$ is $K_{l-1}(W)$ and that

$$\sharp(X(\psi_\beta)/K_{l-1}) = \sharp(X(\psi_\beta)) / (K_{l-1} : K_{l-1}(W)) = q^{n(n+1)/2 - n(n-1)/2} = q^n.$$

On the other hand $\varepsilon \in \mathcal{C}$ also acts on $\psi \in X(\psi_\beta)$ by $\psi^\varepsilon(x) = \psi(\varepsilon x \varepsilon^{-1})$, and let us denote by $X_0(\psi_\beta)$ the set of $\psi \in X(\psi_\beta)$ such that $\psi^\varepsilon = \psi$ for all $\varepsilon \in \mathcal{C}$. Then it is also shown in the proof of Theorem 4.6 of [7] that the set $X_0(\psi_\beta)$ has q^n elements (see Remark 2.1.2 below).

The subgroup of GL_n consisting of the diagonal elements is denoted by D_n . For any $T \in D_n(O)$ we define a one-dimensional representation ω_T of $K_{l-1}(W)$ by

$$\omega_T(x) = \tau(\varpi^{-1} \text{tr}(XT)) = \tau\left(\varpi^{-1} \sum_{i=1}^n X_{ii} T_{ii}\right)$$

for $x = 1_n + \varpi^{l-1} X \pmod{\mathfrak{p}^r} \in K_{l-1}(W)$. Here we denote by A_{ij} the (i, j) -element of a matrix A in general. Then $\omega_T^\varepsilon = \omega_T$ for all $\varepsilon \in \mathcal{C} \pmod{\mathfrak{p}} \subset W$. Furthermore $\omega_T = \omega_{T'}$ for $T, T' \in D_n(O)$ if and only if $T \equiv T'$

(mod \mathfrak{p}). In particular the number of distinct ω_T is q^n . So for any $\psi, \psi' \in X_0(\psi_\beta)$, there exists unique $T \pmod{\mathfrak{p}} \in D_n(\mathbb{F})$ such that $\psi'\psi^{-1} = \omega_T$. Here we put

$$T = \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_f \end{bmatrix}$$

with $T_i \in D_{n_i}(O)$. Then we have the following proposition;

Proposition 2.1.1 ψ and ψ' belong to the same K_{l-1} -orbit in $X(\psi_\beta)$ if and only if $\text{tr } T_i \equiv 0 \pmod{\mathfrak{p}}$ for all $i = 1, \dots, f$.

[Proof] Take any $k = 1_n + \varpi^{l-1}Y \pmod{\mathfrak{p}^r} \in K_{l-1}$. Then

$$k^{-1} = 1_n - \varpi^{l-1}Y + \varpi^{2(l-1)}X^2 \pmod{\mathfrak{p}^r},$$

so that

$$kxk^{-1}x^{-1} = 1_n + \varpi^{2(l-1)}(YX - XY) \pmod{\mathfrak{p}^r} \in K_l$$

for all $x = 1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r} \in K_{l-1}(W)$. Then $\psi' = \psi^k$ if and only if

$$\begin{aligned} \omega_T(x) &= \psi(kxk^{-1}x^{-1}) = \psi_\beta(kxk^{-1}x^{-1}) \\ &= \tau(\varpi^{-1}\text{tr}((YX - XY)\beta)) = \tau(\varpi^{-1}\text{tr}((\beta Y - Y\beta)X)) \end{aligned}$$

for all $x = 1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r} \in K_{l-1}(W)$, that is

$$\text{tr}(XT) \equiv \text{tr}(X(\beta Y - Y\beta)) \pmod{\mathfrak{p}}$$

for all $X \in M_n(O)$ such that $X \pmod{\mathfrak{p}} \in W$. This condition is equivalent to

$$\beta Y - Y\beta \equiv \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_f \end{bmatrix} \pmod{\mathfrak{p}}. \quad (2)$$

Decompose Y into the blocks

$$T = \begin{bmatrix} Y_{11} & \cdots & Y_{1f} \\ \vdots & \ddots & \vdots \\ Y_{f1} & \cdots & Y_{ff} \end{bmatrix}$$

where Y_{ij} is a $n_i \times n_j$ -matrix. Then the condition (2) is equivalent to

$$J_{n_i}(a_i)Y_{ij} - Y_{ij}J_{n_j}(a_j) \equiv \begin{cases} T_i & : i = j, \\ 0 & : i > j \end{cases} \pmod{\mathfrak{p}} \quad (3)$$

for all $i, j = 1, \dots, f$ such that $i \geq j$. This means that $\text{tr}(T_i) \equiv 0 \pmod{\mathfrak{p}}$ for all $i = 1, \dots, f$. After direct calculations, the conditions (3) are equivalent to $Y_{ij} = 0$ if $i > j$ and

$$Y_{ii} \equiv \begin{bmatrix} * & & & \\ t_1 & * & & \\ & t_1 + t_2 & \ddots & \\ & & \ddots & * \\ & & & t_1 + \cdots + t_{n_i-1} & * \end{bmatrix} \pmod{\mathfrak{p}}$$

where t_1, \dots, t_{n_i} are the diagonal elements of T_i , so that $t_1 + \dots + t_{n_i} \equiv 0 \pmod{\mathfrak{p}}$. ■

This proposition shows that the number of K_{l-1} -orbits in $X(\psi_\beta)$ which contains some elements of $X_0(\psi_\beta)$ is q^f , where f is the number of Jordan block in $\bar{\beta}$. This means that the argument in the proof of Theorem 4.6 of [7] works only in the case $f = n$, that is, the case of semi-simple $\bar{\beta}$.

Remark 2.1.2 *Although Hill works over p -adic fields in [7], the arguments in the proof of Theorem 4.6 to show the equations $\sharp(X(\psi_\beta)/K_{l-1}) = q^n$ and $\sharp X_0(\psi_\beta) = q^n$ work well also in the case of equal characteristic local fields.*

Remark 2.1.3 *As the referee points out, Hill's proof fails at $p = 627$ where he claims that $\lambda\psi'_a$ is a $C_G(\hat{a})$ -stable extension of ψ_a to H_a . In fact this is not always the case, because the subgroup N_2/K_l of H_a/K_l is not necessarily normalized by $C_G(\hat{a})$.*

2.2 We present here a proposition which is suggestive for the following arguments.

Proposition 2.2.1 Two elements ψ and ψ' of $X(\psi_\beta)$ belong to the same K_{l-1} -orbit if and only if they have the same restriction on $K_{l-1}(\mathbb{F}[\bar{\beta}])$.

[**Proof**] Assume that $\psi' = \psi^k$ with a $k = 1_n + \varpi^{l-1}Y \pmod{\mathfrak{p}^r} \in K_{l-1}$. Then we have

$$\begin{aligned} \psi'(x)\psi(x)^{-1} &= \psi(kxk^{-1}x^{-1}) = \psi_\beta(kxk^{-1}x^{-1}) \\ &= \tau(\varpi^{-1}\mathrm{tr}((YX - XY)\beta)) = 1 \end{aligned}$$

for all $x = 1_n + \varpi^{l-1} \pmod{\mathfrak{p}^r} \in K_{l-1}(\mathbb{F}[\bar{\beta}])$. The number of elements in the K_{l-1} -orbit of ψ is

$$(K_{l-1} : K_{l-1}(W)) = (M_n(\mathbb{F}) : W) = q^{n(n-1)/2}.$$

On the other hand, the number of elements of $X(\psi_\beta)$ which have the same restriction as ψ on $K_{l-1}(\mathbb{F}[\bar{\beta}])$ is

$$(K_{l-1}(W) : K_{l-1}(\mathbb{F}[\bar{\beta}])) = (W : \mathbb{F}[\bar{\beta}]) = q^{n(n-1)/2}.$$

■

3 General description of the regular characters

In this section, we will describe how Clifford theory works for our problem concerned with $\beta \in M_n(O)$ such that $\bar{\beta} \in M_n(\mathbb{F})$ is regular.

3.1 The arguments of the following sections are based upon an elementary proposition on an irreducible representation of a finite group. For the sake of completeness, we will include here the proposition. Its proof is well-known and can be omitted. See [1, Prop.8.3.3].

Let G be a finite group and N a normal subgroup of G such that G/N is commutative. Let ψ be a group homomorphism of N to \mathbb{C}^\times such that $\psi(gng^{-1}) = \psi(n)$ for all $g \in G$ and $n \in N$. Then the mapping D_ψ of $G/N \times G/N$ to \mathbb{C}^\times is well-defined by

$$D_\psi(\bar{x}, \bar{y}) = \psi([x, y]),$$

where $\bar{x} = x \pmod{N} \in G/N$ and $[x, y] = xyx^{-1}y^{-1}$. In fact we have

$$\begin{aligned} \psi([xn, ym]) &= \psi(xnymn^{-1}x^{-1}m^{-1}y^{-1}) \\ &= \psi(xy(y^{-1}nymn^{-1}x^{-1}m^{-1}x)x^{-1}y^{-1}) \\ &= \psi(xy(y^{-1}nymn^{-1}x^{-1}m^{-1}x)(xy)^{-1}[x, y]) \\ &= \psi(y^{-1}ny)\psi(mn^{-1})\psi(x^{-1}m^{-1}x)\psi([x, y]) = \psi([x, y]) \end{aligned}$$

for all $x, y \in G$ and $m, n \in N$. Then we have $D_\psi(\bar{y}, \bar{x}) = D_\psi(\bar{x}, \bar{y})^{-1}$ and $\bar{x} \mapsto D_\psi(\bar{x}, *)$ is a group homomorphism of G/N to the character group $(G/N)^\wedge$ of the commutative group G/N . Then we have

Proposition 3.1.1 Suppose that D_ψ is non-degenerate, that is, $\bar{x} \mapsto D_\psi(\bar{x}, *)$ is a group isomorphism of G/N onto $(G/N)^\wedge$. Then G has unique irreducible representation π_ψ up to equivalence such that $\langle \psi, \pi_\psi \rangle_N > 0$. In this case

$$\text{Ind}_N^G \psi = \bigoplus^{\dim \pi_\psi} \pi_\psi$$

and $\pi_\psi(n)$ is the homothety $\psi(n)$ for all $n \in N$.

3.2 Fix a $\beta \in M_n(O)$ such that $\bar{\beta} \in M_n(\mathbb{F})$ is regular. The purpose of this subsection is to describe the set $\text{Irr}(G_r \mid \psi_\beta)$ (Theorem 3.2.2) by means of Clifford theory. Since

$$[K_{l-1}(\mathbb{F}[\bar{\beta}]), K_{l-1}(\mathbb{F}[\bar{\beta}))] \subset [K_{l-1}(W), K_{l-1}(W)] \subset \text{Ker}(\psi_\beta)$$

the set $Y(\psi_\beta)$ of the group homomorphisms ψ of $K_{l-1}(\mathbb{F}[\bar{\beta}])$ to \mathbb{C}^\times such that $\psi|_{K_l} = \psi_\beta$ is not empty. Take a $\psi \in Y(\psi_\beta)$.

Define an alternating bilinear form $\langle \cdot, \cdot \rangle_{\bar{\beta}}$ on $M_n(\mathbb{F})$ by

$$\langle X \pmod{\mathfrak{p}}, Y \pmod{\mathfrak{p}} \rangle_{\bar{\beta}} = \text{tr}((XY - YX)\beta) \pmod{\mathfrak{p}}.$$

By virtue of the regularity of $\bar{\beta} \in M_n(\mathbb{F})$, we have

$$\{X \in M_n(\mathbb{F}) \mid \langle X, Y \rangle_{\bar{\beta}} = 0 \text{ for all } Y \in M_n(\mathbb{F})\} = \mathbb{F}[\bar{\beta}].$$

In other word, the \mathbb{F} -vector space $\mathbb{V}_\beta = M_n(\mathbb{F})/\mathbb{F}[\bar{\beta}]$ is a symplectic \mathbb{F} -space with respect to a symplectic form $(\dot{X}, \dot{Y}) \mapsto \langle X, Y \rangle_{\bar{\beta}}$ where $\dot{X} = X \pmod{\mathbb{F}[\bar{\beta}]} \in \mathbb{V}_\beta$ with $X \in M_n(\mathbb{F})$.

We will apply Proposition 3.1.1 to the groups $G = K_{l-1}$ and $N = K_{l-1}(\mathbb{F}[\bar{\beta}])$. First of all, the mapping $1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r} \mapsto X \pmod{\mathfrak{p}}$ induces an isomorphism of G/N onto \mathbb{V}_β . In particular G/N is commutative. Then, for any $x = 1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r} \in K_{l-1}(\mathbb{F}[\bar{\beta}])$ and $k = 1_n + \varpi^{l-1}Y \pmod{\mathfrak{p}^r} \in K_{l-1}$, we have

$$\psi(kxk^{-1})\psi(x)^{-1} = \psi_\beta(kxk^{-1}x^{-1}) = \hat{\tau}(\text{tr}((YX - XY)\beta)) = 1,$$

that is $\psi(kxk^{-1}) = \psi(x)$. Now define $D_\psi(\bar{x}, \bar{y}) = \psi([x, y])$ for $\bar{x}, \bar{y} \in G/N$. If we write

$$x = 1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r}, \quad y = 1_n + \varpi^{l-1}Y \pmod{\mathfrak{p}^r} \in K_{l-1},$$

then we have

$$D_\psi(\bar{x}, \bar{y}) = \psi_\beta(xy x^{-1}y^{-1}) = \tau \left(\varpi^{-1} \langle \bar{X}, \bar{Y} \rangle_{\bar{\beta}} \right),$$

so that D_ψ is non-degenerate. Hence, by Proposition 3.1.1, there exists a unique, up to equivalence, irreducible representation π_ψ of K_{l-1} with representation space H_ψ such that $\langle \psi, \pi_\psi \rangle_{K_{l-1}(\mathbb{F}[\bar{\beta}])} > 0$. Furthermore, for any $x \in K_{l-1}(\mathbb{F}[\bar{\beta}])$, the operator $\pi_\psi(x)$ is the homothety of $\psi(x)$.

Take any $\varepsilon \in \mathcal{C}$. Then for any

$$x = 1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r} \in K_{l-1}(\mathbb{F}[\bar{\beta}])$$

we have

$$\varepsilon x \varepsilon^{-1} x^{-1} \equiv 1_n + \varpi^{l-1}(\varepsilon X \varepsilon^{-1} - X) + \varpi^{2(l-1)}(X^2 - \varepsilon X \varepsilon^{-1} X) \pmod{\mathfrak{p}^r}.$$

Here $x \in K_{l-1}(\mathbb{F}[\bar{\beta}])$ means that $X \pmod{\mathfrak{p}} \in \mathbb{F}[\bar{\beta}]$ so that $\varepsilon X \varepsilon^{-1} \equiv X \pmod{\mathfrak{p}}$. Then $\varepsilon x \varepsilon^{-1} x^{-1} \in K_l$ and we have

$$\begin{aligned} \psi(\varepsilon x \varepsilon^{-1} x^{-1}) &= \psi_\beta(\varepsilon x \varepsilon^{-1} x^{-1}) \\ &= \tau \left(\varpi^{-l} \text{tr}((\varepsilon X \varepsilon^{-1} - X)\beta) + \varpi^{-1} \text{tr}((X^2 - \varepsilon X \varepsilon^{-1} X)\beta) \right) = 1, \end{aligned}$$

that is $\psi(\varepsilon x \varepsilon^{-1}) = \psi(x)$ for all $x \in K_{l-1}(\mathbb{F}[\bar{\beta}])$. Then $\pi_\psi^\varepsilon(k) = \pi_\psi(\varepsilon k \varepsilon^{-1})$ defines an irreducible representation of K_{l-1} on H_ψ equivalent to π_ψ . So there exists a $U(\varepsilon) \in GL_{\mathbb{C}}(H_\psi)$ such that

$$\pi_\psi(\varepsilon k \varepsilon^{-1}) = U(\varepsilon) \circ \pi_\psi(k) \circ U(\varepsilon)^{-1} \quad (4)$$

for all $k \in K_{l-1}$. Let us denote by c_U the two cocycle associated with U , that is $c_U(\varepsilon, \eta) \in \mathbb{C}^\times$ such that

$$U(\varepsilon) \circ U(\eta) = c_U(\varepsilon, \eta) \cdot U(\varepsilon \eta) \quad (5)$$

for all $\varepsilon, \eta \in \mathcal{C}$. Then the cohomology class $c(\psi) \in H^2(\mathcal{C}, \mathbb{C}^\times)$ of c_U is independent of the choice of $U(\varepsilon)$ for each $\varepsilon \in \mathcal{C}$. Here \mathcal{C} acts trivially on \mathbb{C}^\times .

Now we will propose the following hypothesis;

Hypothesis 3.2.1 *The Schur multiplier $c(\psi) \in H^2(\mathcal{C}, \mathbb{C}^\times)$ is trivial for all $\psi \in Y(\psi_\beta)$.*

In section 5, when $\text{ch } \mathbb{F} \neq 2$, we will describe the 2-cocycle c_U much more explicitly (the formula (18)) by means of the Schrödinger representation and the Weil representation associated with the symplectic space \mathbb{V}_β over finite field \mathbb{F} , and will show that

- 1) Hypothesis 3.2.1 is valid if the characteristic polynomial of $\bar{\beta} \in M_n(\mathbb{F})$ is separable (Theorem 5.2.1),

- 2) we have enough evidences to believe that Hypothesis 3.2.1 is valid if $\overline{\beta}$ is split and $\text{ch } \mathbb{F}$ is big enough (Theorem 5.3.1 and the remark after the theorem).

Now we will assume Hypothesis 3.2.1. Then there exists an extension $\tilde{\pi}_\psi$ of π_ψ to $\mathcal{C}K_{l-1}$. In fact, we can assume that U is a group homomorphism of \mathcal{C} to $GL_{\mathbb{C}}(H_\psi)$. For any $\varepsilon \in \mathcal{C} \cap K_{l-1}$, there exists a $\theta(\varepsilon) \in \mathbb{C}^\times$ such that $\pi_\psi(\varepsilon) = \theta(\varepsilon) \cdot U(\varepsilon)$. Then θ is a character of $\mathcal{C} \cap K_{l-1}$ which has an extension Θ to the commutative group \mathcal{C} . Now $\tilde{\pi}_\psi(\varepsilon k) = \Theta(\varepsilon) \cdot U(\varepsilon) \circ \pi_\psi(k)$ ($\varepsilon \in \mathcal{C}, k \in K_{l-1}$) is a well-defined extension of π_ψ to $\mathcal{C}K_{l-1}$.

Our purpose is to determine the set $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$. Take any $\sigma \in \text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$. Then

$$\sigma \hookrightarrow \text{Ind}_{K_l}^{\mathcal{C}K_{l-1}} \psi_\beta = \text{Ind}_{K_{l-1}}^{\mathcal{C}K_{l-1}} \left(\text{Ind}_{K_l}^{K_{l-1}} \psi_\beta \right).$$

We have

$$\text{Ind}_{K_l}^{K_{l-1}} \psi_\beta = \text{Ind}_{K_{l-1}(\mathbb{F}[\overline{\beta}])}^{K_{l-1}} \left(\text{Ind}_{K_l}^{K_{l-1}(\mathbb{F}[\overline{\beta}])} \psi_\beta \right)$$

and

$$\text{Ind}_{K_l}^{K_{l-1}(\mathbb{F}[\overline{\beta}])} \psi_\beta = \bigoplus_{\psi \in Y(\psi_\beta)} \psi$$

because $[K_{l-1}(\mathbb{F}[\overline{\beta}]), K_{l-1}(\mathbb{F}[\overline{\beta}])] \subset \text{Ker}(\psi_\beta)$. For each $\psi \in Y(\psi_\beta)$, we have

$$\text{Ind}_{K_{l-1}(\mathbb{F}[\overline{\beta}])}^{K_{l-1}} \psi = \bigoplus^{\dim \pi_\psi} \pi_\psi$$

by Proposition 3.1.1. Hence there exists a $\psi \in Y(\psi_\beta)$ such that $\sigma \hookrightarrow \text{Ind}_{K_{l-1}}^{\mathcal{C}K_{l-1}} \pi_\psi$. Now we have

$$\text{Ind}_{K_{l-1}}^{\mathcal{C}K_{l-1}} \pi_\psi = \bigoplus_{\chi \in \text{Irr}(\mathcal{C}/\mathcal{C} \cap K_{l-1})} \chi \otimes \tilde{\pi}_\psi$$

where $\text{Irr}(\mathcal{C}/\mathcal{C} \cap K_{l-1})$ is the character group of the abelian group $\mathcal{C}/\mathcal{C} \cap K_{l-1}$ which is identified with the one-dimensional representations of $\mathcal{C}K_{l-1}$ trivial on K_{l-1} . Hence $\{\chi \otimes \tilde{\pi}_\psi\}_\chi$ are the extensions of π_ψ to $\mathcal{C}K_{l-1}$ and $\sigma = \chi \otimes \tilde{\pi}_\psi$ for some $\chi \in \text{Irr}(\mathcal{C}/\mathcal{C} \cap K_{l-1})$. Then the Clifford theory gives

Theorem 3.2.2 *Under the Hypothesis 3.2.1, a bijection*

$$\text{Irr}(\mathcal{C}/\mathcal{C} \cap K_{l-1}) \times Y(\psi_\beta) \rightarrow \text{Irr}(G_r \mid \psi_\beta)$$

is given by $(\chi, \psi) \mapsto \text{Ind}_{\mathcal{C}K_{l-1}}^{G_r} (\chi \otimes \tilde{\pi}_\psi)$.

4 Schur multiplier associated with symplectic space over finite field

In this section the finite field \mathbb{F} is supposed to be of odd characteristic. Our purpose is to define and to study a Schur multiplier associated with a symplectic space over \mathbb{F} . The results of this section are used in the next section to describe the Schur multiplier in Hypothesis 3.2.1. Fix a regular $\overline{\beta} \in M_n(\mathbb{F})$.

4.1 Due to the regularity of $\bar{\beta} \in M_n(\mathbb{F})$, we have

$$\{X \in M_n(\mathbb{F}) \mid X\bar{\beta} = \bar{\beta}X\} = \mathbb{F}[\bar{\beta}].$$

Then $\mathbb{V}_\beta = M_n(\mathbb{F})/\mathbb{F}[\bar{\beta}]$ is a symplectic space over \mathbb{F} with respect to the symplectic form

$$\langle \dot{X}, \dot{Y} \rangle_{\bar{\beta}} = \text{tr}((XY - YX)\bar{\beta}).$$

Here we denote $\dot{X} = X \pmod{\mathbb{F}[\bar{\beta}]} \in \mathbb{V}_\beta$ for $X \in M_n(\mathbb{F})$. Take a character $\rho : \mathbb{F}[\bar{\beta}] \rightarrow \mathbb{C}^\times$ of the additive group $\mathbb{F}[\bar{\beta}]$. Let $v \mapsto [v]$ be a \mathbb{F} -linear splitting of the exact sequence

$$0 \rightarrow \mathbb{F}[\bar{\beta}] \rightarrow M_n(\mathbb{F}) \rightarrow \mathbb{V}_\beta \rightarrow 0 \quad (6)$$

of \mathbb{F} -vector space. In other word let us choose a \mathbb{F} -vector subspace $V \subset M_n(\mathbb{F})$ such that $M_n(\mathbb{F}) = V \oplus \mathbb{F}[\bar{\beta}]$, and put $v = [v] \pmod{\mathbb{F}[\bar{\beta}]}$ with $[v] \in V$ for $v \in \mathbb{V}_\beta$. For any $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$ and $v = \dot{X} \in \mathbb{V}_\beta$, put $\varepsilon^{-1}v\varepsilon = \varepsilon^{-1}X\varepsilon \pmod{\mathbb{F}[\bar{\beta}]}$ in \mathbb{V}_β and

$$\gamma(v, \varepsilon) = \varepsilon^{-1}[v]\varepsilon - [\varepsilon^{-1}v\varepsilon] \in \mathbb{F}[\bar{\beta}].$$

Since $v \mapsto \rho(\gamma(v, \varepsilon))$ is an additive character of \mathbb{V}_β , there exists uniquely a $v_\varepsilon \in \mathbb{V}_\beta$ such that

$$\rho(\gamma(v, \varepsilon)) = \hat{\tau}(\langle v, v_\varepsilon \rangle_{\bar{\beta}})$$

for all $v \in \mathbb{V}_\beta$. Then, for any $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$, we have

$$v_{\varepsilon\eta} = \varepsilon v_\eta \varepsilon^{-1} + v_\varepsilon \quad (7)$$

because

$$\gamma(v, \varepsilon\eta) = \gamma(v, \varepsilon) + \gamma(\varepsilon^{-1}v\varepsilon, \eta)$$

for all $v \in \mathbb{V}_\beta$. Put

$$c_{\bar{\beta}, \rho}(\varepsilon, \eta) = \hat{\tau}(2^{-1}\langle v_\varepsilon, v_{\varepsilon\eta} \rangle_{\bar{\beta}})$$

for $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$. Then the relation (7) shows that $c_{\bar{\beta}, \rho} \in Z^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$ is a 2-cocycle with trivial action of $\mathbb{F}[\bar{\beta}]^\times$ on \mathbb{C}^\times . Furthermore we have

Proposition 4.1.1 *The cohomology class $[c_{\bar{\beta}, \rho}] \in H^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$ is independent of the choice of the \mathbb{F} -linear splitting $v \mapsto [v]$.*

[Proof] Take another \mathbb{F} -linear splitting $v \mapsto [v]'$ with respect to which we will define $\gamma'(v, \varepsilon) \in \mathbb{F}[\bar{\beta}]$ and $v'_\varepsilon \in \mathbb{V}_\beta$ as above. Then there exists a $\delta \in \mathbb{V}_\beta$ such that $\rho([v] - [v]') = \hat{\tau}(\langle v, \delta \rangle_{\bar{\beta}})$ for all $v \in \mathbb{V}_\beta$. We have $v'_\varepsilon = v_\varepsilon + \delta - \varepsilon^{-1}\delta\varepsilon$ for all $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$. So if we put

$$\alpha(\varepsilon) = \hat{\tau}(2^{-1}\langle v'_\varepsilon - v_{\varepsilon^{-1}}, \delta \rangle_{\bar{\beta}})$$

for $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$, then we have

$$\hat{\tau}(2^{-1}\langle v'_\varepsilon, v'_{\varepsilon\eta} \rangle_{\bar{\beta}}) = \hat{\tau}(2^{-1}\langle v_\varepsilon, v_{\varepsilon\eta} \rangle_{\bar{\beta}}) \cdot \alpha(\eta)\alpha(\varepsilon\eta)^{-1}\alpha(\varepsilon)$$

for all $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$. ■

Let $H(\mathbb{V}_\beta) = \mathbb{V}_\beta \times \mathbb{C}^1$ be the Heisenberg group associated with the symplectic \mathbb{F} -space \mathbb{V}_β . Here \mathbb{C}^1 is the multiplicative group of the complex number with absolute value one. The group operation on $H(\mathbb{V}_\beta)$ is defined by

$$(u, s) \cdot (v, t) = (u + v, s \cdot t \cdot \widehat{\tau} \left(2^{-1} \langle u, v \rangle_{\overline{\beta}} \right)).$$

There exists uniquely up to isomorphism an irreducible representation $(\pi_{\overline{\beta}}, H_{\overline{\beta}})$ of $H(\mathbb{V}_\beta)$ such that $\pi_{\overline{\beta}}(0, s) = s$ for all $(0, s) \in H(\mathbb{V}_\beta)$. We call the representation $\pi_{\overline{\beta}}$ the Schrödinger representation of the Heisenberg group $H(\mathbb{V}_\beta)$. For any $\sigma \in Sp(\mathbb{V}_\beta)$, the mapping $(u, s) \mapsto (u\sigma, s)$ is an automorphism of $H(\mathbb{V}_\beta)$ so that there exists a $T(\sigma) \in GL_{\mathbb{C}}(H_{\overline{\beta}})$ such that

$$\pi_{\overline{\beta}}(u\sigma, s) = T(\sigma)^{-1} \circ \pi_{\overline{\beta}}(u, s) \circ T(\sigma)$$

for all $(u, s) \in H(\mathbb{V}_\beta)$. For any $\sigma, \sigma' \in Sp(\mathbb{V}_\beta)$, there exists a $c_T(\sigma, \sigma') \in \mathbb{C}^\times$ such that

$$T(\sigma) \circ T(\sigma') = c_T(\sigma, \sigma') \cdot T(\sigma\sigma').$$

Then $c_T \in Z^2(Sp(\mathbb{V}_\beta), \mathbb{C}^\times)$ is a 2-cocycle with trivial action of $Sp(\mathbb{V}_\beta)$ on \mathbb{C}^\times , and the cohomology class $[c_T] \in H^2(Sp(\mathbb{V}_\beta), \mathbb{C}^\times)$ is independent of the choice of each $T(\sigma)$. We have a group homomorphism $\varepsilon \mapsto \sigma_\varepsilon$ of $\mathbb{F}[\overline{\beta}]^\times$ to $Sp(\mathbb{V}_\beta)$ defined by $\sigma_\varepsilon(v) = \varepsilon^{-1}v\varepsilon$. So a 2-cocycle $c_T \in Z^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ and a cohomology class $[c_T] \in H^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ is defined by $c_T(\varepsilon, \eta) = c_T(\sigma_\varepsilon, \sigma_\eta)$. Put

$$U(\varepsilon) = \pi_{\overline{\beta}}(v_\varepsilon, 0) \circ T(\sigma_\varepsilon) \in GL_{\mathbb{C}}(H_{\overline{\beta}})$$

for $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$. Then (7) gives

$$U(\varepsilon) \circ U(\eta) = c_{\overline{\beta}, \rho}(\varepsilon, \eta) \cdot c_T(\varepsilon, \eta) \cdot U(\varepsilon\eta)$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$. Now we will investigate the validity of the following hypothesis

Hypothesis 4.1.2 *The cohomology class $[c_{\overline{\beta}, \rho} c_T] \in H^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ is trivial for all additive characters ρ of $\mathbb{F}[\overline{\beta}]$.*

Remark 4.1.3 *If ρ is trivial, then $c_{\overline{\beta}, \rho}(\varepsilon, \eta) = 1$ for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$. So Hypothesis 4.1.2 asserts that the cohomology class $[c_T] \in H^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ is trivial and that the cohomology classes $[c_{\overline{\beta}, \rho}] \in H^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ are trivial for all ρ .*

4.2 The Schrödinger representation of the Heisenberg group $H(\mathbb{V}_\beta)$ is realized as follows. Let $\mathbb{V}_\beta = \mathbb{W}' \oplus \mathbb{W}$ be a polarization of the symplectic \mathbb{F} -space \mathbb{V}_β . Let $L^2(\mathbb{W}')$ be the complex vector space of the complex-valued functions f on \mathbb{W}' with norm $|f|^2 = \sum_{w \in \mathbb{W}'} |f(w)|^2$. The Schrödinger representation $\pi_{\overline{\beta}}$ of $H(\mathbb{V}_\beta)$ is realized on $L^2(\mathbb{W}')$ by

$$(\pi_{\overline{\beta}}(u, s)f)(w) = s \cdot \widehat{\tau} \left(2^{-1} \langle u_-, u_+ \rangle_{\overline{\beta}} + \langle w, u_+ \rangle_{\overline{\beta}} \right) \cdot f(w + u_-)$$

for $(u, s) \in H(\mathbb{V}_\beta)$ where $u = u_- + u_+$ with $u_- \in \mathbb{W}'$, $u_+ \in \mathbb{W}$. We call it the Schrödinger representation associated with the polarization $\mathbb{V}_\beta = \mathbb{W}' \oplus \mathbb{W}$.

The method of [8] and [9] enable us to give an explicit description of 2-cocycle $c_T \in Z^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$. Each element $\sigma \in Sp(\mathbb{V}_\beta)$ is denoted by blocks $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the polarization $\mathbb{V}_\beta = \mathbb{W}' \oplus \mathbb{W}$ (that is $a \in \text{End}_{\mathbb{F}}(\mathbb{W}')$, $b \in \text{Hom}_{\mathbb{F}}(\mathbb{W}', \mathbb{W})$ etc). Then a unitary operator $T_{\hat{\tau}}(\sigma)$ of $L^2(\mathbb{W}')$ is defined by

$$(T_{\hat{\tau}}(\sigma)f)(w) = \text{const.} \times \sum_{v \in \mathbb{W}/\text{Ker } c} f(wa + vc) \cdot \hat{\tau} \left(2^{-1} \langle wa + vc, wb + vd \rangle_{\bar{\beta}} - 2^{-1} \langle w, v \rangle_{\bar{\beta}} \right)$$

with a positive real number const. and we have

$$\pi_{\hat{\tau}}(u\sigma, s) = T_{\hat{\tau}}(\sigma)^{-1} \circ \pi_{\hat{\tau}}(u, s) \circ T_{\hat{\tau}}(\sigma).$$

Defined in [8] and [9] is the generalized Weil constant $\gamma_{\hat{\tau}}(Q) \in \mathbb{C}^\times$ associated with a \mathbb{F} -quadratic form Q having the following properties;

- 1) if Q is an orthogonal sum of two \mathbb{F} -quadratic forms Q_1 and Q_2 , then $\gamma_{\hat{\tau}}(Q) = \gamma_{\hat{\tau}}(Q_1) \cdot \gamma_{\hat{\tau}}(Q_2)$,
- 2) $\gamma_{\hat{\tau}}(0) = 1$, and if Q is a regular \mathbb{F} -quadratic form on a \mathbb{F} -vector space X

$$\gamma_{\hat{\tau}}(Q) = q^{-2^{-1} \dim_{\mathbb{F}} X} \sum_{x \in X} \hat{\tau}(Q(x)).$$

Particularly for $a \in \mathbb{F}^\times$, we have

$$\gamma_{\hat{\tau}}(a) = q^{-1/2} \sum_{x \in \mathbb{F}} \hat{\tau}(ax^2) = \left(\frac{a}{\mathbb{F}} \right) \cdot q^{-1/2} \sum_{x \in \mathbb{F}^\times} \left(\frac{x}{\mathbb{F}} \right) \cdot \hat{\tau}(x)$$

where

$$\left(\frac{a}{\mathbb{F}} \right) = \begin{cases} 1 & : \text{if } a \text{ is a square in } \mathbb{F}, \\ -1 & : \text{if } a \text{ is not a square in } \mathbb{F} \end{cases}$$

is the Legendre symbol with respect to the field \mathbb{F} . Now for every $\sigma, \sigma' \in Sp(\mathbb{V}_\beta)$, let us denote by $Q_{\sigma, \sigma'}$ the \mathbb{F} -quadratic form on $\mathbb{W}' \times \mathbb{W}' \times \mathbb{W}'$ defined by

$$(u, v, w) \mapsto \langle u, v\sigma' \rangle_{\bar{\beta}} + \langle v, w\sigma \rangle_{\bar{\beta}} - \langle u, w\sigma\sigma' \rangle_{\bar{\beta}}.$$

Put $c_{\hat{\tau}}(\sigma, \sigma') = \gamma_{\hat{\tau}}(Q_{\sigma, \sigma'})^{-1}$, then [8] and [9] give the formula

$$T_{\hat{\tau}}(\sigma) \circ T_{\hat{\tau}}(\sigma') = c_{\hat{\tau}}(\sigma, \sigma') \cdot T_{\hat{\tau}}(\sigma\sigma').$$

So if we put $T(\varepsilon) = T_{\hat{\tau}}(\sigma_\varepsilon)$ for $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$, then we have

$$c_T(\varepsilon, \eta) = c_{\hat{\tau}}(\sigma_\varepsilon, \sigma_\eta) = \gamma_{\hat{\tau}}(Q_{\varepsilon, \eta})^{-1}$$

for $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$. Here we put $Q_{\varepsilon, \eta} = Q_{\sigma_\varepsilon, \sigma_\eta}$. Put

$$B(\mathbb{V}_\beta) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Sp(\mathbb{V}_\beta) \right\}$$

which is a subgroup of $Sp(\mathbb{V}_\beta)$. For any $\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbb{V}_\beta)$, we have

$$(T_{\hat{\tau}}(\sigma)f)(w) = \hat{\tau} \left(2^{-1} \langle wa, wb \rangle_{\bar{\beta}} \right) \cdot f(wa)$$

for $f \in L^2(\mathbb{W}')$, and $\sigma \mapsto T_{\hat{\tau}}(\sigma)$ is a group homomorphism of $B(\mathbb{V}_\beta)$ in $GL_{\mathbb{C}}(L^2(\mathbb{W}'))$. Hence $c_{\hat{\tau}}(\sigma, \sigma') = 1$ for all $\sigma, \sigma' \in B(\mathbb{V}_\beta)$. In other word, if $\mathbb{W}\sigma_\varepsilon = \mathbb{W}, \mathbb{W}\sigma_\eta = \mathbb{W}$ for $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$, then $c_T(\varepsilon, \eta) = 1$.

4.3 Let $\chi_{\bar{\beta}}(t) = \prod_{i=1}^r p_i(t)^{e_i}$ be the irreducible decomposition of the characteristic polynomial $\chi_{\bar{\beta}}(t)$ of $\bar{\beta}$ with distinct irreducible polynomials $p_i(t)$ in $\mathbb{F}[t]$ ($e_i > 0$). So we can assume that $\bar{\beta}$ is of the form

$$\bar{\beta} = \begin{bmatrix} \bar{\beta}_1 & & \\ & \ddots & \\ & & \bar{\beta}_r \end{bmatrix}$$

where $\bar{\beta}_i \in M_{n_i}(t)$ is a regular element with characteristic polynomial $p_i(t)^{e_i}$, so that $n_i = e_i \deg p_i(t)$. Then we have

$$\mathbb{F}[\bar{\beta}] = \left\{ \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_r \end{bmatrix} \mid X_i \in \mathbb{F}[\bar{\beta}_i] \right\}.$$

Define \mathbb{F} -vector subspaces L_{β}, M_{β} of $M_n(\mathbb{F})$ by

$$L_{\beta} = \left\{ \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_r \end{bmatrix} \mid X_i \in M_{n_i}(\mathbb{F}) \right\}$$

and

$$M_{\beta} = \left\{ \begin{bmatrix} O_{n_1} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & O_{n_r} \end{bmatrix} \in M_n(\mathbb{F}) \right\}$$

respectively. Then we have

$$\mathbb{V}_{\beta} = L_{\beta}/\mathbb{F}[\bar{\beta}] \oplus M_{\beta} = \bigoplus_{i=1}^r \mathbb{V}_{\beta_i} \oplus M_{\beta}.$$

Here M_{β} is a symplectic \mathbb{F} -space with respect to the symplectic form $\langle X, Y \rangle_{\bar{\beta}} = \text{tr}((XY - YX)\bar{\beta})$ with a polarization $M_{\beta} = M_{\beta}^{-} \oplus M_{\beta}^{+}$ defined by

$$M_{\beta}^{+} = \left\{ \begin{bmatrix} O_{n_1} & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & O_{n_r} \end{bmatrix} \in M_n(\mathbb{F}) \right\},$$

and

$$M_{\beta}^{-} = \left\{ \begin{bmatrix} O_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & O_{n_r} \end{bmatrix} \in M_n(\mathbb{F}) \right\}.$$

Let $(\pi_{\bar{\beta}, M}, L^2(M_{\beta}^{-}))$ be the Schrödinger representation of the Heisenberg group $H(M_{\beta}) = M_{\beta} \times \mathbb{C}^1$ associated with the polarization $M_{\beta} = M_{\beta}^{-} \oplus M_{\beta}^{+}$. Any $\varepsilon \in \mathbb{F}[\bar{\beta}]^{\times}$ defines an element $X \mapsto \varepsilon^{-1} X \varepsilon$ of $Sp(M_{\beta})$ which keeps the subspace M_{β}^{\pm} stable. So define a $T_M(\varepsilon) \in GL_{\mathbb{C}}(L^2(M_{\beta}^{-}))$ by

$$(T_M(\varepsilon)f)(W) = f(\varepsilon^{-1}W\varepsilon).$$

Then $\varepsilon \mapsto T_M(\varepsilon)$ is a group homomorphism of $\mathbb{F}[\overline{\beta}]^\times$ such that

$$\pi_{\overline{\beta},M}(\varepsilon^{-1}X\varepsilon, s) = T_M(\varepsilon)^{-1} \circ \pi_{\overline{\beta},M}(X, s) \circ T_M(\varepsilon)$$

for all $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$ and $(X, s) \in H(M_\beta)$.

The symplectic form $\langle \cdot, \cdot \rangle_{\overline{\beta}}$ on \mathbb{V}_β induces the symplectic form $\langle \cdot, \cdot \rangle_{\overline{\beta}_i}$ on \mathbb{V}_{β_i} . Let us denote $(\pi_{\overline{\beta}_i}, H_i)$ the Schrödinger representation of the Heisenberg group $H(\mathbb{V}_{\beta_i})$, and choose $T_i(\varepsilon) \in GL_{\mathbb{C}}(H_i)$ for each $\varepsilon \in \mathbb{F}[\overline{\beta}_i]^\times$ such that

$$\pi_{\overline{\beta}_i}(\varepsilon^{-1}v\varepsilon, s) = T_i(\varepsilon)^{-1} \circ \pi_{\overline{\beta}_i}(v, s) \circ T_i(\varepsilon)$$

for all $(v, s) \in H(\mathbb{V}_{\beta_i})$. Let $c_i \in Z^2(\mathbb{F}[\overline{\beta}_i]^\times, \mathbb{C}^\times)$ be the 2-cocycle associated with T_i , that is

$$T_i(\varepsilon) \circ T_i(\eta) = c_i(\varepsilon, \eta) \cdot T_i(\varepsilon\eta)$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}_i]^\times$. Now we have a surjective group homomorphism

$$((X, s), (v_1, s_1), \dots, (v_r, s_r)) \mapsto \left(X + \sum_{i=1}^r v_i, s \cdot \prod_{i=1}^r s_i \right)$$

of $H(M_\beta) \times \prod_{i=1}^r H(\mathbb{V}_{\beta_i})$ onto $H(\mathbb{V}_\beta)$, and the tensor product $\pi_{\overline{\beta},M} \otimes \bigotimes_{i=1}^r \pi_{\overline{\beta}_i}$, which is trivial on the kernel of the group homomorphism, induces the Schrödinger representation of $H(\mathbb{V}_\beta)$ on the space $L^2(M_\beta^-) \otimes \bigotimes_{i=1}^r H_i$. Put

$$T(\varepsilon) = T_M(\varepsilon) \otimes \bigotimes_{i=1}^r T_i(\varepsilon_i)$$

for $\varepsilon = \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_r \end{bmatrix} \in \mathbb{F}[\overline{\beta}]^\times$, and we have

$$c_T(\varepsilon, \eta) = \prod_{i=1}^r c_i(\varepsilon_i, \eta_i) \tag{8}$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$.

Let us assume that the \mathbb{F} -vector subspace V of $M_n(\mathbb{F})$ associated with the \mathbb{F} -linear splitting $v \mapsto [v]$ contains M_β . Then we have $[\varepsilon] \in L_\beta$ for all $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$ and

$$c_{\overline{\beta},\rho}(\varepsilon, \eta) = \prod_{i=1}^r c_{\overline{\beta}_i,\rho}(\varepsilon_i, \eta_i) \tag{9}$$

for all $\varepsilon = \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_r \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_r \end{bmatrix} \in \mathbb{F}[\overline{\beta}]^\times$.

Proposition 4.3.1 *Let a group homomorphism $\tilde{\rho} : L_\beta \rightarrow \mathbb{C}^\times$ be an extension of $\rho : \mathbb{F}[\bar{\beta}] \rightarrow \mathbb{C}^\times$. Then we have*

$$c_{\bar{\beta}, \rho}(\varepsilon, \eta) = \tilde{\rho}(2^{-1}\varepsilon[v_\eta]\varepsilon^{-1}) \cdot \tilde{\rho}(2^{-1}[v_{\varepsilon\eta}])^{-1} \cdot \tilde{\rho}(2^{-1}[v_\varepsilon])$$

for all $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$.

[Proof] The relation (7) gives $\varepsilon^{-1}v_\varepsilon\varepsilon = -v_{\varepsilon^{-1}}$ so that we have

$$\begin{aligned} c_{\bar{\beta}, \rho}(\varepsilon, \eta) &= \hat{\tau} \left(2^{-1} \langle v_\varepsilon, \varepsilon v_\eta \varepsilon \rangle_{\bar{\beta}} \right) = \hat{\tau} \left(2^{-1} \langle v_\eta, v_{\varepsilon^{-1}} \rangle_{\bar{\beta}} \right) \\ &= \rho \left(2^{-1} \gamma(v_\eta, \varepsilon^{-1}) \right) \\ &= \tilde{\rho} \left(2^{-1} \{ \varepsilon[v_\eta]\varepsilon^{-1} - [v_{\varepsilon\eta} - v_\varepsilon] \} \right). \end{aligned}$$

■

Theorem 4.3.2 *The hypothesis 4.1.2 is valid if the characteristic polynomial of $\beta \in M_n(\mathbb{F})$ is separable.*

[Proof] The formula (8) and (9) enable us to assume that the characteristic polynomial of $\bar{\beta} \in M_n(\mathbb{F})$ is irreducible over \mathbb{F} . In this case $\mathbb{F}[\bar{\beta}]$ is a finite field so that the multiplicative group $\mathbb{F}[\bar{\beta}]^\times$ is a cyclic group. Then $H^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$ is well-known to be trivial. ■

Due to the formula (8) and (9), the hypothesis 4.1.2 is reduced to the case $\chi_{\bar{\beta}}(t) = p(t)^e$ with an irreducible polynomial $p(t) \in \mathbb{F}[t]$. In the following subsections, we will consider two extreme cases

- 1) $e = 1$ (in subsections 4.4 and 4.5), and
- 2) $\deg p(t) = 1$ (in subsection 4.6)

to see how the 2-cocycles $c_{\bar{\beta}, \rho}$ and c_T are written as coboundaries.

4.4 In this subsection, we will assume that the characteristic polynomial $\chi_{\bar{\beta}}(t)$ of $\bar{\beta}$ is irreducible over \mathbb{F} . Then, as is pointed out in the proof of Theorem 4.3.2, the group of the Schur multipliers $H^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$ is trivial. In the following subsections, we will try to express the 2-cocycles $c_{\bar{\beta}, \rho}$ and c_T as coboundaries as explicit as possible.

There exists a \mathbb{F} -basis $\{u_1, \dots, u_n\}$ of $\mathbb{F}[\bar{\beta}]$ such that the regular representation of the field $\mathbb{F}[\bar{\beta}]$ with respect to the \mathbb{F} -basis is the inclusion mapping of $\mathbb{F}[\bar{\beta}]$ into $M_n(\mathbb{F})$. So the trace of $\mathbb{F}[\bar{\beta}]$ over \mathbb{F} is $T_{\mathbb{F}[\bar{\beta}]/\mathbb{F}}(x) = \text{tr}(x)$ for all $x \in \mathbb{F}[\bar{\beta}]$. Then we have a group isomorphism $a \mapsto \rho_a$ of $\mathbb{F}[\bar{\beta}]$ onto the group of additive character of $\mathbb{F}[\bar{\beta}]$ defined by $\rho_a(x) = \hat{\tau}(\text{tr}(ax))$. So $\rho = \rho_a$ has a extension $\tilde{\rho}(x) = \text{tr}(ax)$ to $L_\beta = M_n(\mathbb{F})$ which is $\mathbb{F}[\bar{\beta}]^\times$ -invariant. Then Proposition 4.3.1 shows that the 2-cocycle $c_{\bar{\beta}, \rho} \in Z^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$ is a coboundary

$$c_{\bar{\beta}, \rho}(\varepsilon, \eta) = \gamma(\eta) \cdot \gamma(\varepsilon\eta)^{-1} \gamma(\varepsilon)$$

where $\gamma(\varepsilon) = \hat{\tau}(2^{-1}\text{tr}(a[v_\varepsilon]))$ for $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$.

We will apply the general setting of subsection 4.2 to give an explicit description of 2-cocycle $c_T \in Z^2(\mathbb{F}[\bar{\beta}]^\times, \mathbb{C}^\times)$ by defining a canonical polarization of \mathbb{V}_β . To begin with there exists a symmetric $g \in GL_n(\mathbb{F})$ such that

${}^t\bar{\beta} = g\bar{\beta}g^{-1}$ (for example $g = (\text{tr}(u_i u_j))_{i,j=1,\dots,n}$). Then the \mathbb{F} -linear endomorphism $X \mapsto X^* = g^{-1} {}^t X g$ is an involution of $M_n(\mathbb{F})$, that is, $(X^*)^* = X$ for all $X \in M_n(\mathbb{F})$.

Remark 4.4.1 *We have canonical \mathbb{F} -linear isomorphisms*

$$\mathbb{F}[\bar{\beta}] \otimes_{\mathbb{F}} \mathbb{F}[\bar{\beta}] \xrightarrow{\sim} \text{End}_{\mathbb{F}}(\mathbb{F}[\bar{\beta}]) \xrightarrow{\sim} M_n(\mathbb{F})$$

where the first isomorphism is given by $a \otimes b \mapsto f_{a,b}$ with $f_{a,b}(x) = \text{tr}(bx) \cdot a$ for $x \in \mathbb{F}[\bar{\beta}]$, and the second isomorphism is to take the representation matrix with respect to $\{u_1, \dots, u_n\}$. Then the involution $X \mapsto X^* = g^{-1} {}^t X g$ of $M_n(\mathbb{F})$ induces the \mathbb{F} -linear endomorphism $a \otimes b \mapsto b \otimes a$ of $\mathbb{F}[\bar{\beta}] \otimes_{\mathbb{F}} \mathbb{F}[\bar{\beta}]$.

Put

$$W_{\pm} = \{X \in M_n(\mathbb{F}) \mid X^* = \pm X\}.$$

Then $M_n(\mathbb{F}) = W_- \oplus W_+$ and $\mathbb{F}[\bar{\beta}] \subset W_+$. Let us denote by \mathbb{W}_{\pm} the image of W_{\pm} by the canonical surjection $M_n(\mathbb{F}) \rightarrow \mathbb{V}_{\beta}$. Then $\mathbb{V}_{\beta} = \mathbb{W}_- \oplus \mathbb{W}_+$ is a polarization of the symplectic \mathbb{F} -space \mathbb{V}_{β} because we have $\text{tr}(XY\bar{\beta}) = \text{tr}(Y^*X^*\bar{\beta})$ for all $X, Y \in M_n(\mathbb{F})$. Using this polarization, we have

$$c_T(\varepsilon, \eta) = \gamma_{\hat{\tau}}(Q_{\varepsilon, \eta})^{-1}$$

for $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^{\times}$ where $\gamma_{\hat{\tau}}(Q_{\varepsilon, \eta})$ is the generalized Weil constant of the \mathbb{F} -quadratic form defined on $\mathbb{W}_- \times \mathbb{W}_- \times \mathbb{W}_-$ by

$$Q_{\varepsilon, \eta}(x, y, z) = \langle x, y\sigma_{\eta} \rangle_{\bar{\beta}} + \langle y, z\sigma_{\varepsilon} \rangle_{\bar{\beta}} - \langle x, z\sigma_{\varepsilon\eta} \rangle_{\bar{\beta}}.$$

Our problem is to find an explicit function $\delta : \mathbb{F}[\bar{\beta}]^{\times} \rightarrow \mathbb{C}^{\times}$ such that

$$c_T(\varepsilon, \eta) = \delta(\eta) \cdot \delta(\varepsilon\eta)^{-1} \delta(\varepsilon)$$

for all $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^{\times}$. As we will see in the next subsection, the solution is quite interesting even in the simplest case $n = 2$.

4.5 Suppose that the characteristic polynomial of $\bar{\beta} \in M_2(\mathbb{F})$ is irreducible over \mathbb{F} , and use the notations of the preceding subsection. We can assume that $\bar{\beta} = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$ where $\alpha \in \mathbb{F}^{\times}$ is not a square. Then we have ${}^t\bar{\beta} = g\bar{\beta}g^{-1}$ with $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$W_+ = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in M_2(\mathbb{F}) \right\}, \quad W_- = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \in M_2(\mathbb{F}) \right\}.$$

We will identify \mathbb{W}_+ and \mathbb{W}_- with \mathbb{F} by means of the mappings

$$y \mapsto \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \pmod{\mathbb{F}[\bar{\beta}]}, \quad x \mapsto \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \pmod{\mathbb{F}[\bar{\beta}]}$$

respectively. Then $Sp(\mathbb{V}_{\beta}) = SL_2(\mathbb{F})$ by the block description with respect to the polarization $\mathbb{V}_{\beta} = \mathbb{W}_- \oplus \mathbb{W}_+$. For $\zeta = \begin{bmatrix} a & \alpha b \\ b & a \end{bmatrix} \in \mathbb{F}[\bar{\beta}]$, put

$$\bar{\zeta} = \begin{bmatrix} a & -\alpha b \\ -b & a \end{bmatrix} \in \mathbb{F}[\bar{\beta}], \quad \zeta_+ = a, \quad \zeta_- = b$$

(a collision of notation with $\overline{\beta} = \beta \pmod{\mathfrak{p}}$ occurs here, but there may be no risk of confusion in the following arguments). Then for $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$, we have

$$\sigma_\varepsilon = \begin{pmatrix} (\varepsilon/\overline{\varepsilon})_+ & 2\alpha(\varepsilon/\overline{\varepsilon})_- \\ 2^{-1}(\varepsilon/\overline{\varepsilon})_- & (\varepsilon/\overline{\varepsilon})_+ \end{pmatrix} \in SL_2(\mathbb{F}).$$

Proposition 4.5.1 *For $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$, we have*

- 1) *if $(\varepsilon/\overline{\varepsilon})_- \cdot (\eta/\overline{\eta})_- \cdot (\varepsilon\eta/\overline{\varepsilon\eta})_- = 0$, then $c_{\hat{\tau}}(\varepsilon, \eta) = 1$,*
- 2) *if $(\varepsilon/\overline{\varepsilon})_- \cdot (\eta/\overline{\eta})_- \cdot (\varepsilon\eta/\overline{\varepsilon\eta})_- \neq 0$, then*

$$c_{\hat{\tau}}(\varepsilon, \eta) = \gamma_{\hat{\tau}}(\alpha(\varepsilon/\overline{\varepsilon})_-) \cdot \gamma_{\hat{\tau}}(\alpha(\eta/\overline{\eta})_-) \cdot \gamma_{\hat{\tau}}(\alpha(\varepsilon\eta/\overline{\varepsilon\eta})_-)^{-1}.$$

[Proof] The quadratic form $Q_{\varepsilon, \eta}$ on $\mathbb{W}_- \times \mathbb{W}_- \times \mathbb{W}_- = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ is

$$Q_{\varepsilon, \eta}(x, y, z) = 4\alpha(\eta/\overline{\eta})_- \cdot xy + 4\alpha(\varepsilon/\overline{\varepsilon})_- \cdot yz - 4\alpha(\varepsilon\eta/\overline{\varepsilon\eta})_- \cdot zx$$

and the symmetric matrix associated with it is

$$\tilde{Q}_{\varepsilon, \eta} = \begin{bmatrix} 0 & 4\alpha(\eta/\overline{\eta})_- & -4\alpha(\varepsilon\eta/\overline{\varepsilon\eta})_- \\ 4\alpha(\eta/\overline{\eta})_- & 0 & 4\alpha(\varepsilon/\overline{\varepsilon})_- \\ -4\alpha(\varepsilon\eta/\overline{\varepsilon\eta})_- & 4\alpha(\varepsilon/\overline{\varepsilon})_- & 0 \end{bmatrix}.$$

If $(\varepsilon/\overline{\varepsilon})_- = 0$, then $\delta = \varepsilon/\overline{\varepsilon} = \pm 1$ and

$$P\tilde{Q}_{\varepsilon, \eta}^t P = \begin{bmatrix} 0 & 8\delta\alpha(\eta/\overline{\eta})_- & 0 \\ 8\delta\alpha(\eta/\overline{\eta})_- & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta & -1 \\ 0 & \delta & 1 \end{bmatrix}$. Hence the quadratic form $Q_{\varepsilon, \eta}$ is equivalent to 0 or the orthogonal sum of 0 and a hyperbolic plane, and we have

$$c_{\hat{\tau}}(\varepsilon, \eta) = \gamma_{\hat{\tau}}(Q_{\varepsilon, \eta})^{-1} = 1.$$

The cases $(\eta/\overline{\eta})_- = 0$ or $(\varepsilon\eta/\overline{\varepsilon\eta})_- = 0$ is treated similarly.

If $(\varepsilon/\overline{\varepsilon})_- (\eta/\overline{\eta})_- (\varepsilon\eta/\overline{\varepsilon\eta})_- \neq 0$, then $Q_{\varepsilon, \eta}$ is a regular quadratic form on a finite field \mathbb{F} of odd characteristic. Then $Q_{\varepsilon, \eta}$ is equivalent to the quadratic form

$$(x, y, z) \mapsto 4\alpha(\varepsilon/\overline{\varepsilon})_- \cdot x^2 + 4\alpha(\eta/\overline{\eta})_- \cdot y^2 - 4\alpha(\varepsilon\eta/\overline{\varepsilon\eta})_- \cdot z^2$$

due to [10, p.39, Th.3.8]. Hence we have

$$\begin{aligned} c_{\hat{\tau}}(\varepsilon, \eta) &= \gamma_{\hat{\tau}}(Q_{\varepsilon, \eta}) \\ &= \gamma_{\hat{\tau}}(4\alpha(\varepsilon/\overline{\varepsilon})_-) \gamma_{\hat{\tau}}(4\alpha(\eta/\overline{\eta})_-) \gamma_{\hat{\tau}}(4\alpha(\varepsilon\eta/\overline{\varepsilon\eta})_-)^{-1}. \end{aligned}$$

■

Finally we have an explicit expression of $c_{\hat{\tau}}$ as a coboundary

Theorem 4.5.2 Put

$$\delta(\varepsilon) = \begin{cases} \gamma_{\hat{\tau}}(\alpha(\varepsilon/\bar{\varepsilon})_-) & : (\varepsilon/\bar{\varepsilon})_- \neq 0, \\ \left(\frac{\varepsilon/\bar{\varepsilon}}{\mathbb{F}}\right) & : (\varepsilon/\bar{\varepsilon})_- = 0 \end{cases}$$

for $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$. Then we have

$$c_{\hat{\tau}}(\varepsilon, \eta) = \delta(\eta)\delta(\varepsilon\eta)^{-1}\delta(\varepsilon)$$

for all $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$.

4.6 In this subsection, we will assume that the characteristic polynomial of $\bar{\beta} \in M_n(\mathbb{F})$ is $\chi_{\bar{\beta}}(t) = (t - a)^n$ with $a \in \mathbb{F}$. In this case we can suppose that

$$\bar{\beta} = J_n(a) = \begin{bmatrix} a & 1 & & \\ & a & \ddots & \\ & & \ddots & 1 \\ & & & a \end{bmatrix}$$

is a Jordan block. Let W_+ and W_- be the \mathbb{F} -vector subspace of $M_n(\mathbb{F})$ consisting of the upper triangular matrices and the lower triangular matrices with the diagonal elements 0 respectively. Let \mathbb{W}_\pm be the image of W_\pm with respect to the canonical surjection $M_n(\mathbb{F}) \rightarrow \mathbb{V}_\beta$. Then $\mathbb{V}_\beta = \mathbb{W}_- \oplus \mathbb{W}_+$ is a polarization of the symplectic \mathbb{F} -space \mathbb{V}_β . Note that $\mathbb{W}_+ \sigma_\varepsilon = \mathbb{W}_+$ for any $\varepsilon \in \mathbb{F}[\bar{\beta}]^\times$. Hence $c_T(\varepsilon, \eta) = 1$ for all $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$ as described in subsection 4.2.

Until the end of this subsection we will assume that the characteristic of \mathbb{F} is greater than n . Then we have a group isomorphism

$$\mathbb{F}^\times \times \mathbb{F}^{n-1} \xrightarrow{\sim} \mathbb{F}[\bar{\beta}]^\times$$

defined by $(r, s_1, \dots, s_{n-1}) \mapsto r \cdot \exp\left(\sum_{k=1}^{n-1} s_k J_n(0)^k\right)$. Here

$$\exp S = \sum_{k=0}^{n-1} \frac{1}{k!} S^k$$

is the exponential of upper triangular matrix $S \in M_n(\mathbb{F})$ with diagonal elements 0. For any $A \in \mathbb{F}[\bar{\beta}]$, put $\rho_A(X) = \text{tr}(X^t A)$ ($X \in \mathbb{F}[\bar{\beta}]$). Then $A \mapsto \rho_A$ gives a group isomorphism of $\mathbb{F}[\bar{\beta}]$ onto the group of the additive characters of $\mathbb{F}[\bar{\beta}]$. Let V be the \mathbb{F} -vector space consisting of the $X \in M_n(\mathbb{F})$ such that $\text{tr}(X^t A) = 0$ for all $A \in \mathbb{F}[\bar{\beta}]$. Then we have $M_n(\mathbb{F}) = V \oplus \mathbb{F}[\bar{\beta}]$ because $\text{ch } \mathbb{F} > n$, and a \mathbb{F} -linear splitting of the exact sequence (6) is defined with respect to this V . Now any additive character $\rho = \rho_A$ of $\mathbb{F}[\bar{\beta}]$ ($A \in \mathbb{F}[\bar{\beta}]$) has an extension $\tilde{\rho}(X) = \text{tr}(X^t A)$ to $L_\beta = M_n(\mathbb{F})$, and Proposition 4.3.1 gives

$$c_{\bar{\beta}, \rho}(\varepsilon, \eta) = \hat{\tau}(2^{-1} \text{tr}(\varepsilon[v_\eta] \varepsilon^{-1} {}^t A))$$

for all $\varepsilon, \eta \in \mathbb{F}[\bar{\beta}]^\times$.

Proposition 4.6.1 *An additive character $\rho = \rho_A$ of $\mathbb{F}[\overline{\beta}]$ with $A \in \mathbb{F}[\overline{\beta}]$ has an extension $\tilde{\rho}$ to $M_n(\mathbb{F})$ as an additive character such that $\tilde{\rho}(\varepsilon X \varepsilon^{-1}) = \tilde{\rho}(X)$ for all $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$ if and only if $A \in \mathbb{F}[\overline{\beta}]$ is a diagonal matrix. In this case we have $c_{\overline{\beta}, \rho}(\varepsilon, \eta) = 1$ for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$.*

[Proof] Every extensions $\tilde{\rho}$ of ρ to $M_n(\mathbb{F})$ is written in the form

$$\tilde{\rho}(X) = \hat{\tau}(\text{tr}(X {}^t A)) \cdot \hat{\tau}(\langle X, X_0 \rangle_{\overline{\beta}})$$

with certain $X_0 \in M_n(\mathbb{F})$. So $\tilde{\rho}(\varepsilon X \varepsilon^{-1}) = \tilde{\rho}(X)$ for all $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$ means that $B = {}^t A + X_0 \overline{\beta} - \overline{\beta} X_0$ is commutative with any $\varepsilon \in \mathbb{F}[\overline{\beta}]^\times$. Since $c 1_n - \overline{\beta} \in \mathbb{F}[\overline{\beta}]^\times$ for a suitable $c \in \mathbb{F}$, the matrix ${}^t A + X_0 \overline{\beta} - \overline{\beta} X_0$ is commutative with $\overline{\beta}$. Hence $B \in \mathbb{F}[\overline{\beta}]$ and

$$\rho(X) = \tilde{\rho}(X) = \hat{\tau}(\text{tr}(XB))$$

for all $X \in \mathbb{F}[\overline{\beta}]$. This means that A is a diagonal matrix. In this case we have

$$c_{\overline{\beta}, \rho}(\varepsilon, \eta) = \hat{\tau}(2^{-1} \text{tr}([v_\eta] {}^t A)) = 1$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$. ■

We can calculate the 2-cocycle $c_{\overline{\beta}, \rho}$ explicitly for small n . Take an additive character $\rho = \rho_A$ of $\mathbb{F}[\overline{\beta}]$ with

$$A = \sum_{k=0}^{n-1} \rho_k J_n(0)^k \in \mathbb{F}[\overline{\beta}].$$

For the calculation of $c_{\overline{\beta}, \rho}(\varepsilon, \eta)$, there is no loss of generality if we assume that ε and η are of the form $\exp S$ with an upper triangular $S \in \mathbb{F}[\overline{\beta}] \subset M_n(\mathbb{F})$ with diagonal elements 0. In the following examples, we will give explicit formulas for $c_{\overline{\beta}, \rho}$ in the cases $n = 2, 3, 4$.

Example 4.6.2 *For $\varepsilon = \exp\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \eta = \exp\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in \mathbb{F}[\overline{\beta}]^\times$, we have*

$$c_{\overline{\beta}, \eta}(\varepsilon, \eta) = \hat{\tau}(2^{-1} \rho_1^2 \cdot (r^2 u + r u^2)).$$

If $\text{ch } \mathbb{F} > 3$, then put

$$\delta(\varepsilon) = \hat{\tau}\left(-\frac{1}{2 \cdot 3} \rho_1^2 \cdot r^3\right)$$

and we have

$$c_{\overline{\beta}, \eta}(\varepsilon, \eta) = \delta(\eta) \delta(\varepsilon \eta)^{-1} \delta(\varepsilon)$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$.

Example 4.6.3 *If $\text{ch } \mathbb{F} > 5$, put*

$$\delta(\varepsilon) = \hat{\tau}\left(-\frac{1}{2} \left\{ \frac{1}{3} \rho_1^2 \cdot r^3 + 2 \rho_1 \rho_2 \cdot r^2 s + \rho_2^2 \left(r s^2 - \frac{1}{2^2 \cdot 5} r^5 \right) \right\}\right)$$

for $\varepsilon = \exp\left(\begin{bmatrix} 0 & r & s \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}\right) \in \mathbb{F}[\overline{\beta}]^\times$. Then we have

$$c_{\overline{\beta},\rho}(\varepsilon, \eta) = \delta(\eta)\delta(\varepsilon\eta)^{-1}\delta(\varepsilon)$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$.

Example 4.6.4 If $\text{ch } \mathbb{F} > 7$, put

$$\delta(\varepsilon) = \widehat{\tau} \left[-\frac{1}{2} \begin{pmatrix} \frac{1}{3}\rho_1^2 \cdot r^3 + 2\rho_1\rho_2 \cdot r^2s + \rho_2^2 \cdot \left(2rs^2 + r^2t - \frac{1}{2 \cdot 3 \cdot 5}r^5\right) \\ + \rho_1\rho_3 \cdot \left(2rs^2 + 2r^2t + \frac{1}{2 \cdot 3 \cdot 5}r^5\right) \\ + \rho_2\rho_3 \cdot \left(4rst + \frac{4}{3}s^3 - \frac{1}{3}r^4s\right) \\ + \rho_3^2 \cdot \left(s^2t + rt^2 + \frac{1}{2^2 \cdot 3}r^4t - \frac{1}{3}r^3s^2 + \frac{1}{2^2 \cdot 3^2 \cdot 7}r^7\right) \end{pmatrix} \right]$$

for $\varepsilon = \exp\left(\begin{bmatrix} 0 & r & s & t \\ 0 & 0 & r & s \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \in \mathbb{F}[\overline{\beta}]^\times$. Then we have

$$c_{\overline{\beta},\rho}(\varepsilon, \eta) = \delta(\eta)\delta(\varepsilon\eta)^{-1}\delta(\varepsilon)$$

for all $\varepsilon, \eta \in \mathbb{F}[\overline{\beta}]^\times$.

These examples strongly suggest that the following conjecture is valid for all $n \geq 2$.

Conjecture 4.6.5 The Schur multiplier $[c_{\overline{\beta},\rho}] \in H^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ is trivial for all additive characters ρ of $\mathbb{F}[\overline{\beta}]$ if the characteristic of \mathbb{F} is big enough.

Remark 4.6.6 It may be an interesting problem to find a regular $\overline{\beta} \in M_n(\mathbb{F})$ whose characteristic polynomial is a power of an irreducible polynomial over \mathbb{F} of degree greater than or equal to 2 and an additive character ρ of $\mathbb{F}[\overline{\beta}]$ such that the Schur multiplier $[c_{\overline{\beta},\rho}] \in H^2(\mathbb{F}[\overline{\beta}]^\times, \mathbb{C}^\times)$ is not trivial.

5 Construction of the regular characters via Weil representation

In this section we will construct the irreducible representation π_ψ of subsection 3.2 by means of Schrödinger representation of Heisenberg group and Weil representation over finite field. We will also describe the Schur multiplier of Hypothesis 3.2.1 by means of the Schur multiplier defined in Proposition 4.1.1.

Throughout this section the finite field \mathbb{F} is supposed to be of odd characteristic, although the arguments of subsection 5.1 work well without this assumption.

5.1 Let us determine the 2-cocycle of the group extension

$$0 \rightarrow M_n(O_{l-1}) \rightarrow K_{l-1} \rightarrow M_n(\mathbb{F}) \rightarrow 0 \quad (10)$$

defined by the isomorphism $K_l \xrightarrow{\sim} M_n(O_{l-1})$ ($1_n + \varpi^l X \pmod{\mathfrak{p}^r} \mapsto X \pmod{\mathfrak{p}^{l-1}}$) and $K_{l-1}/K_l \xrightarrow{\sim} M_n(\mathbb{F})$ induced by $1_n + \varpi^{l-1} X \pmod{\mathfrak{p}^r} \mapsto X \pmod{\mathfrak{p}}$. Fix a mapping $\lambda : M_n(\mathbb{F}) \rightarrow M_n(O)$ such that $X = \lambda(X) \pmod{\mathfrak{p}}$ for all $X \in M_n(\mathbb{F})$ and $\lambda(0) = 0$, and define a mapping $l : M_n(\mathbb{F}) \rightarrow K_{l-1}$ by $X \mapsto 1_n + \varpi^{l-1} \lambda(X) \pmod{\mathfrak{p}^r}$. Then, for any $k = 1_n + \varpi^l S \pmod{\mathfrak{p}^r} \in K_l$, we have

$$\begin{aligned} l(X)kl(X)^{-1} &= (1_n + \varpi^{l-1} \lambda(X))(1_n + \varpi^l S)(1_n - \varpi^{l-1} \lambda(X) + \varpi^{2(l-1)} \lambda(X)^2) \pmod{\mathfrak{p}^r} \\ &= 1_n + \varpi^l S \pmod{\mathfrak{p}^r}, \end{aligned}$$

and

$$l(X)l(Y)l(X+Y)^{-1} = 1_n + \varpi^l (\varpi^{l-2} \lambda(X)\lambda(Y) + \mu(X, Y)) \pmod{\mathfrak{p}^r}$$

for all $X, Y \in M_n(\mathbb{F})$, where $\mu : M_n(\mathbb{F}) \times M_n(\mathbb{F}) \rightarrow M_n(O)$ is defined by

$$\lambda(X) + \lambda(Y) - \lambda(X+Y) = \varpi \mu(X, Y).$$

So the 2-cocycle of the group extension (10) is

$$[(\overline{X}, \widehat{Y}) \mapsto \varpi^{l-2} XY + \mu(\overline{X}, \widehat{Y}) \pmod{\mathfrak{p}^{l-1}}] \in Z^2(M_n(\mathbb{F}), M_n(O_{l-1}))$$

with the trivial action of $M_n(\mathbb{F})$ on $M_n(O_{l-1})$, where $\overline{X} = X \pmod{\mathfrak{p}} \in M_n(\mathbb{F})$. Now we have two 2-cocycles

$$\begin{aligned} c &= [(\overline{X}, \widehat{Y}) \mapsto \varpi^{l-2} XY \pmod{\mathfrak{p}^{l-1}}] \in Z^2(M_n(\mathbb{F}), M_n(O_{l-1})) \text{ and} \\ \mu &= [(X, Y) \mapsto \mu(X, Y) \pmod{\mathfrak{p}^{l-1}}] \in Z^2(M_n(\mathbb{F}), M_n(O_{l-1})). \end{aligned}$$

Let us denote by \mathbb{G} and \mathbb{M} the groups associated with the 2-cocycles c and μ respectively. More precisely, the group operation on $\mathbb{G} = M_n(\mathbb{F}) \times M_n(O_{l-1})$ is defined by

$$(\overline{X}, \overline{S}) \cdot (\widehat{Y}, \overline{T}) = ((X+Y)^\wedge, S+T + \varpi^{l-2} XY \pmod{\mathfrak{p}^{l-1}})$$

and the group operation on $\mathbb{M} = M_n(\mathbb{F}) \times M_n(O_{l-1})$ is defined by

$$(X, \overline{S}) \cdot (Y, \overline{T}) = (X+Y, S+T + \mu(X, Y) \pmod{\mathfrak{p}^{l-1}}).$$

The fiber product of \mathbb{G} and \mathbb{M} with respect to the projections $\mathbb{G} \rightarrow M_n(\mathbb{F})$ and $\mathbb{M} \rightarrow M_n(\mathbb{F})$ is denoted by $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$. That is $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$ is the subgroup of the direct product $\mathbb{G} \times \mathbb{M}$ consisting of the elements

$$(X; S, S') = ((X, S), (X, S')).$$

Then we have a surjective group homomorphism $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M} \rightarrow K_{l-1}$

$$(X; \overline{S}, \overline{S'}) \mapsto 1_n + \varpi^{l-1} \lambda(X) + \varpi^l (S + S') \pmod{\mathfrak{p}^r} \quad (11)$$

whose kernel $\langle \mathbb{G}, \mathbb{M} \rangle$ consists of the elements $(0; S, -S)$. A morphism

$$\mathrm{tr}_\beta : M_n(O_{l-1}) \rightarrow O_{l-1} \quad (X \pmod{\mathfrak{p}^{l-1}} \mapsto \mathrm{tr}(X\beta) \pmod{\mathfrak{p}^{l-1}})$$

of additive groups induces a morphism

$$\mathrm{tr}_\beta^* : H^2(M_n(\mathbb{F}), M_n(O_{l-1})) \rightarrow H^2(M_n(\mathbb{F}), O_{l-1})$$

of cohomology groups. Here $M_n(\mathbb{F})$ acts on O_{l-1} trivially. The group associated with the 2-cocycle $c_\beta = c \circ \mathrm{tr}_\beta \in Z^2(M_n(\mathbb{F}), O_{l-1})$ is $\mathcal{H}_\beta = M_n(\mathbb{F}) \times O_{l-1}$ with group operation

$$(\overline{X}, s) \cdot (\overline{Y}, t) = ((X + Y)^\wedge, s + t + \varpi^{l-2} \mathrm{tr}(XY\beta) \pmod{\mathfrak{p}^{l-1}}).$$

We have a surjective group homomorphism

$$\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M} \rightarrow \mathcal{H}_\beta \quad ((X; \overline{S}, \overline{S'}) \mapsto (X, \overline{\mathrm{tr}(S\beta)})) \quad (12)$$

whose kernel is

$$\{(0; \overline{S}, \overline{S'}) \mid \mathrm{tr}(S\beta) \equiv 0 \pmod{\mathfrak{p}^{l-1}}\}. \quad (13)$$

The center of \mathcal{H}_β is $Z(\mathcal{H}_\beta) = \mathbb{F}[\overline{\beta}] \times O_{l-1}$, in fact $(X, s) \in Z(\mathcal{H}_\beta)$ is equivalent to $\langle X, Y \rangle_{\overline{\beta}} = 0$ for all $Y \in M_n(\mathbb{F})$, which is equivalent to $X \in \mathbb{F}[\overline{\beta}]$. Let us denote by $(\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M})(\mathbb{F}[\overline{\beta}])$ the inverse image of the center $Z(\mathcal{H}_\beta)$ by the surjection (12). Then $(\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M})(\mathbb{F}[\overline{\beta}])$ is projected onto $K_{l-1}(\mathbb{F}[\overline{\beta}])$ by the surjection (11). We have a group homomorphism

$$\psi_0 : \mathbb{M} \rightarrow \mathbb{C}^\times \quad ((X, \overline{S'}) \mapsto \tau(\varpi^{-l} \mathrm{tr}(\lambda(X) + \varpi S')\beta))$$

which induces a group homomorphism $\tilde{\psi}_0 : \mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M} \rightarrow \mathbb{C}^\times$ via the projection $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M} \rightarrow \mathbb{M}$.

Take any $\psi \in Y(\psi_\beta)$. Combining with the projection (11), we define a group homomorphism $\tilde{\psi} : (\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M})(\mathbb{F}[\overline{\beta}]) \rightarrow \mathbb{C}^\times$. Then $\tilde{\psi}_0^{-1} \cdot \tilde{\psi}$ induces a group homomorphism ρ of $Z(\mathcal{H}_\beta)$ to \mathbb{C}^\times because $\tilde{\psi}_0^{-1} \cdot \tilde{\psi}$ is trivial on the kernel (13). The inverse image of K_l under the surjection (11) is projected onto $O_{l-1} \subset Z(\mathcal{H}_\beta)$ by the surjection (12), and $\psi|_{K_l} = \psi_\beta$ means that $\rho|_{O_{l-1}} = \psi_1$ where

$$\psi_1 : O_{l-1} \rightarrow \mathbb{C}^\times \quad (s \pmod{\mathfrak{p}^{l-1}} \mapsto \tau(\varpi^{-(l-1)} s))$$

Thus we have a bijection of $Y(\psi_\beta)$ onto the subset of the character group of $Z(\mathcal{H}_\beta)$ consisting of the extensions of ψ_1 .

Let us consider the action of $\mathcal{C} = (O_r[\beta_r])^\times$ on the groups $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$ and \mathcal{H}_β . For any $(\overline{X}; \overline{S}, \overline{S'}) \in \mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$ and $\varepsilon \in O[\beta]^\times$, we have

$$\begin{aligned} \varepsilon^{-1}(1_n + \varpi^{l-1} \lambda(\overline{X}) + \varpi^l(S + S'))\varepsilon \\ = 1_n + \varpi^{l-1} \lambda((\varepsilon^{-1} X \varepsilon)^\wedge) + \varpi^l(\varepsilon^{-1}(S + S')\varepsilon + \nu(\overline{X}, \varepsilon)) \end{aligned}$$

where $\nu : M_n(\mathbb{F}) \times \mathcal{C} \rightarrow M_n(O)$ is defined by

$$\varepsilon^{-1} \lambda(X) \varepsilon - \lambda(\varepsilon^{-1} X \varepsilon) = \varpi \cdot \nu(X, \varepsilon).$$

Then $\overline{\varepsilon} \in \mathcal{C}$ acts on $(\overline{X}; \overline{S}, \overline{S'}) \in \mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$ by

$$(\overline{X}; \overline{S}, \overline{S'})^\varepsilon = ((\varepsilon^{-1} X \varepsilon)^\wedge; \overline{\varepsilon^{-1} S \varepsilon}, \overline{\varepsilon^{-1} S' \varepsilon + \nu(\overline{X}, \varepsilon)}),$$

and on $(\overline{X}, s) \in \mathcal{H}_\beta$ by $(\overline{X}, s)^\varepsilon = ((\varepsilon^{-1} X \varepsilon)^\wedge, s)$. We have

$$\tilde{\psi}_0((X; S, S')^\varepsilon) = \tilde{\psi}_0(X; S, S').$$

5.2 Let us modify the group structure on \mathcal{H}_β into a form more suitable for further examinations. The coboundary of a mapping

$$\delta : M_n(\mathbb{F}) \rightarrow O_{l-1} \quad (X \pmod{\mathfrak{p}} \mapsto 2^{-1}\varpi^{l-2}\mathrm{tr}(X^2\beta) \pmod{\mathfrak{p}^{l-1}})$$

is

$$\begin{aligned} \partial\delta(\overline{X}, \hat{Y}) &= \delta(\hat{Y}) - \delta((X+Y)^\wedge) + \delta(\overline{X}) \\ &\equiv -2^{-1}\varpi^{l-s}\mathrm{tr}((XY+YX)\beta) \pmod{\mathfrak{p}^{l-1}} \end{aligned}$$

so that we will redefine the group operation on $\mathcal{H}_\beta = M_n(\mathbb{F}) \times O_{l-1}$ to be associated with the 2-cocycle $c_\beta + \partial\beta$, that is

$$(\overline{X}, s) \cdot (\hat{Y}, t) = ((X+Y)^\wedge, s+t + \overline{2^{-1}\varpi^{l-2}\mathrm{tr}((XY-YX)\beta)}).$$

Under this group operation, the center \mathcal{H}_β is $Z(\mathcal{H}_\beta) = \mathbb{F}[\overline{\beta}] \times O_{l-1}$, and for any $(X, s) \in Z(\mathcal{H}_\beta)$ and $(Y, t) \in \mathcal{H}_\beta$ we have

$$(X, s) \cdot (Y, t) = (X+Y, s+t).$$

In particular $Z(\mathcal{H}_\beta)$ is the direct product of additive groups $\mathbb{F}[\overline{\beta}]$ and O_{l-1} . The surjection (12) is now

$$(\overline{X}; \overline{S}, \overline{S'}) \mapsto (\overline{X}, \overline{\mathrm{tr}(S\beta) - 2^{-1}\varpi^{l-2}\mathrm{tr}(X^2\beta)}). \quad (14)$$

We have a central extension

$$0 \rightarrow Z(\mathcal{H}_\beta) \rightarrow \mathcal{H}_\beta \xrightarrow{(*)} \mathbb{V}_\beta \rightarrow 0 \quad (15)$$

where $(*) : (X, s) \mapsto \dot{X} = X \pmod{\mathbb{F}[\overline{\beta}]}$. Let us determine the 2-cocycle of this group extension. Fix an \mathbb{F} -vector subspace $V_\beta \subset M_n(\mathbb{F})$ such that $M_n(\mathbb{F}) = V_\beta \oplus \mathbb{F}[\overline{\beta}]$, and let $[x] \in V_\beta$ be the representative of $x \in \mathbb{V}_\beta$. Put

$$l : \mathbb{V}_\beta \rightarrow \mathcal{H}_\beta. \quad (x \mapsto ([x], 0))$$

Then for any $X, Y \in \mathbb{V}_\beta$ we have

$$l(x)l(y)l(x+y)^{-1} = (0, 2^{-1}\varpi^{l-2}\mathrm{tr}((XY-YX)\beta) \pmod{\mathfrak{p}^{l-1}})$$

where $[x] = \overline{X}, [y] = \overline{Y}$ with $X, Y \in M_n(O)$. So the 2-cocycle of the central extension (15) is

$$[(\dot{\overline{X}}, \dot{\hat{Y}}) \mapsto (0, 2^{-1}\varpi^{l-2}\mathrm{tr}((XY-YX)\beta) \pmod{\mathfrak{p}^{l-1}})] \in Z^2(\mathbb{V}_\beta, Z(\mathcal{H}_\beta)).$$

The group operation on $\mathbb{H}_\beta = \mathbb{V}_\beta \times Z(\mathcal{H}_\beta)$ defined by this 2-cocycle is

$$(x, s) \cdot (y, t) = (x+y, s+t + (0, 2^{-1}\varpi^{l-2}\mathrm{tr}((XY-YX)\beta) \pmod{\mathfrak{p}^{l-1}}))$$

where $x = \dot{\overline{X}} \in \mathbb{V}_\beta$ with $X \in M_n(O)$ etc. The group \mathbb{H}_β is isomorphic to \mathcal{H}_β by $(x, (Y, s)) \mapsto ([x] + Y, s)$.

Let $\mathbb{V}_\beta = \mathbb{W}' \oplus \mathbb{W}$ be a polarization of the symplectic \mathbb{F} -space \mathbb{V}_β . Then we have defined in the subsection 4.2 the Schrödinger representation $(\pi_\beta, L^2(\mathbb{W}'))$

of the Heisenberg group $H(\mathbb{V}_\beta)$ associated with the polarization. We have also defined, for each $\sigma \in Sp(\mathbb{V}_\beta)$, an element $T_{\hat{\tau}}(\sigma) \in GL_{\mathbb{C}}(L^2(\mathbb{W}'))$ such that

$$\pi_\beta(u\sigma, s) = T_{\hat{\tau}}(\sigma)^{-1} \circ \pi_\beta(u, s) \circ T_{\hat{\tau}}(\sigma)$$

for all $(u, s) \in H(\mathbb{V}_\beta)$.

Take a $\psi \in Y(\psi_\beta)$. Then, as described in the preceding section, the group homomorphism $\tilde{\psi}_0^{-1} \cdot \tilde{\psi}$ corresponds to a group homomorphism $\rho \otimes \psi_1$ of $Z(\mathcal{H}_\beta) = \mathbb{F}[\bar{\beta}] \times O_{l-1}$ where ρ is a group homomorphism of the additive group $\mathbb{F}[\bar{\beta}]$ to \mathbb{C}^\times . More explicitly

$$\rho(\bar{X}) = \tau(2^{-1}\varpi^{-1}\text{tr}(X^2\beta) - \varpi^{-l}\text{tr}(\lambda(\bar{X})\beta)) \cdot \psi(1_n + \varpi^{l-1}\lambda(\bar{X}))$$

for $\bar{X} \in \mathbb{F}[\bar{\beta}]$, or

$$\psi(\bar{g}) = \rho(\bar{X}) \cdot \tau(\varpi^{-l}\text{tr}(X\beta) - 2^{-1}\varpi^{-1}\text{tr}(X^2\beta)) \quad (16)$$

for $\bar{g} = 1_n + \varpi^{l-1}X \pmod{\mathfrak{p}^r} \in K_{l-1}(\mathbb{F}[\bar{\beta}])$. Then the irreducible representation $(\pi_\beta, L^2(\mathbb{W}'))$ of $H(\mathbb{V}_\beta)$ combined with the isomorphism $\mathcal{H}_\beta \xrightarrow{\sim} \mathbb{H}_\beta$ and the group homomorphism

$$\mathbb{H}_\beta \rightarrow H(\mathbb{V}_\beta) \quad ((v, z) \mapsto (v, \rho \otimes \psi_1(z)))$$

defines an irreducible representation $(\pi_{\beta,\rho}, L^2(\mathbb{W}'))$ of \mathcal{H}_β such that $\pi_{\beta,\rho}(z) = \rho \otimes \psi_1(z)$ for all $z \in Z(\mathcal{H}_\beta)$. Then $\pi_{\beta,\rho}$ combined with the surjection (14) defines an irreducible representation $(\tilde{\pi}_{\beta,\rho}, L^2(\mathbb{W}'))$ of $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$ such that $\tilde{\pi}_{\beta,\rho}(g) = \tilde{\psi}_0^{-1} \cdot \tilde{\psi}(g)$ for all $g \in (\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M})(\mathbb{F}[\bar{\beta}])$. Because $\tilde{\psi}_0 \cdot \tilde{\pi}_{\beta,\rho}$ is trivial on $\langle \mathbb{G}, \mathbb{M} \rangle$, the representation $\tilde{\psi}_0 \cdot \tilde{\pi}_{\beta,\rho}$ of $\mathbb{G} \times_{M_n(\mathbb{F})} \mathbb{M}$ induces an irreducible representation $(\pi_{\beta,\psi}, L^2(\mathbb{W}'))$ of K_{l-1} such that $\pi_{\beta,\psi}(k) = \psi(k)$ for all $k \in K_{l-1}(\mathbb{F}[\bar{\beta}])$.

Let us consider the action of \mathcal{C} on \mathbb{H}_β and on $\pi_{\beta,\rho}$. Take an $\varepsilon \in \mathcal{C}$. Let us denote by $\bar{\varepsilon} \in \mathbb{F}[\bar{\beta}]^\times$ the image of $\varepsilon \in \mathcal{C}$ by the canonical surjection $\mathcal{C} \rightarrow \mathbb{F}[\bar{\beta}]^\times$. Because $\sigma_{\bar{\varepsilon}} \in Sp(\mathbb{V}_\beta)$, put

$$T(\varepsilon) = T_{\hat{\tau}}(\sigma_{\bar{\varepsilon}}) \in GL_{\mathbb{C}}(L^2(\mathbb{W}')).$$

Then, with the notations of the subsection 4.1

$$T(\varepsilon) \circ T(\eta) = c_T(\bar{\varepsilon}, \bar{\eta}) \cdot T(\varepsilon\eta)$$

for all $\varepsilon, \eta \in \mathcal{C}$. On the other hand, for any $(x, (Y, s)) \in \mathbb{H}_\beta$, we have

$$([x] + Y, s)^\varepsilon = ([\bar{\varepsilon}^{-1}x\bar{\varepsilon}] + Y + \gamma(x, \bar{\varepsilon}), s)$$

where we use the notations of subsection 4.1. So $\varepsilon \in \mathcal{C}$ acts on $(x, (Y, s)) \in \mathbb{H}_\beta$ by

$$\begin{aligned} (x, (Y, s))^\varepsilon &= (\bar{\varepsilon}^{-1}x\bar{\varepsilon}, (Y + \gamma(x, \bar{\varepsilon}), s)) \\ &= (x\sigma_{\bar{\varepsilon}}, (Y, s)) \cdot (0, (\gamma(x, \bar{\varepsilon}), 0)). \end{aligned}$$

Then we have

$$\pi_{\beta,\rho}((X, s)^\varepsilon) = \rho(\gamma([\dot{X}], \bar{\varepsilon})) \cdot T(\varepsilon)^{-1} \circ \pi_{\beta,\rho}(X, s) \circ T(\varepsilon)$$

for all $(X, s) \in \mathcal{H}_\beta$. We have, with the notations of subsection 4.1,

$$\rho(\gamma(x, \bar{\varepsilon})) = \hat{\tau} \left(\langle x, v_{\bar{\varepsilon}} \rangle_{\bar{\beta}} \right)$$

for all $x \in \mathbb{V}_\beta$. Since

$$(v_\beta, 1)^{-1}(x, s)(v_\beta, 1) = (x, s \cdot \tau \left(\varpi^{-1} \langle x, v_\beta \rangle_{\bar{\beta}} \right))$$

for all $(x, s) \in H(\mathbb{V}_\beta)$, we have

$$\pi_{\beta, \rho}((X, s)^\varepsilon) = U(\varepsilon)^{-1} \circ \pi_{\beta, \rho}(X, s) \circ U(\varepsilon)$$

for all $(X, s) \in \mathcal{H}_\beta$ where we put

$$U(\varepsilon) = T(\varepsilon) \circ \pi_\beta(v_\beta, 1) \in GL_{\mathbb{C}}(L^2(\mathbb{W}')).$$

Then we have

$$\pi_{\beta, \psi}(\varepsilon^{-1} k \varepsilon) = U(\varepsilon)^{-1} \circ \pi_{\beta, \psi}(k) \circ U(\varepsilon)$$

for all $k \in K_{l-1}$ and

$$U(\varepsilon) \circ U(\eta) = c_{\bar{\beta}, \rho}(\bar{\varepsilon}, \bar{\eta}) c_T(\bar{\varepsilon}, \bar{\varepsilon}) \cdot U(\varepsilon \eta) \quad (17)$$

for all $\varepsilon, \eta \in \mathcal{C}$. Furthermore $U(\varepsilon) = 1$ for all $\varepsilon \in \mathcal{C} \cap K_{l-1}$ because $\mathcal{C} \cap K_{l-1} \subset K_{l-1}(\mathbb{F}[\bar{\beta}])$ and the action of $\varepsilon \in \mathcal{C} \cap K_{l-1}$ on \mathcal{H}_β is trivial.

Now we have constructed explicitly $U(\varepsilon)$ of the formula (4), and comparing (5) with (17), we have

$$c_U(\varepsilon, \eta) = c_{\bar{\beta}, \rho}(\bar{\varepsilon}, \bar{\eta}) \cdot c_T(\bar{\varepsilon}, \bar{\eta}) \quad (18)$$

for all $\varepsilon, \eta \in \mathcal{C}$. Then Theorem 4.3.2 shows

Theorem 5.2.1 *Hypothesis 3.2.1 is valid if the characteristic polynomial of $\bar{\beta} \in M_n(\mathbb{F})$ is separable.*

5.3 In this subsection, we will assume that $\bar{\beta} \in M_n(\mathbb{F})$ is regular split and that $\bar{\beta}$ is in the Jordan canonical form (1). Assume also that the characteristic of \mathbb{F} is odd. Then the formula (18) is based upon the canonical polarization $\mathbb{V}_\beta = \mathbb{W}_- \oplus \mathbb{W}_+$ given at the beginning of the subsection 4.6. In this case we have $c_T(\varepsilon, \eta) = 1$ for all $\varepsilon, \eta \in \mathcal{C}$.

Now we will reconsider the results of section 2. For a diagonal matrix A in $\mathbb{F}[\bar{\beta}]$, define an element $\psi_A \in X_0(\psi_\beta)$ by

$$\psi_A(g) = \tau \left(\varpi^{-l} \text{tr}(X\beta) - 2^{-1} \varpi^{-1} \text{tr}(X^2\beta) + \varpi^{-1} \text{tr}(XA) \right)$$

for $g = \overline{1_n + \varpi^{l-1} X} \in K_{l-1}(W)$. Then $\psi = \psi_A|_{K_{l-1}(\mathbb{F}[\bar{\beta}])} \in Y(\psi_\beta)$ corresponds to the additive character $\rho = \rho_A$ of $\mathbb{F}[\bar{\beta}]$ by (16). For such a $\psi \in Y(\psi_\beta)$, we have $c(\psi) = 1$ in $H^2(\mathcal{C}, \mathbb{C}^\times)$ by Proposition 4.6.1 and by (18). On the other hand the number of the diagonal elements in $\mathbb{F}[\bar{\beta}]$ is q^r , and the remark just after Proposition 2.1.1 shows that the characters ψ_A with diagonal matrix $A \in \mathbb{F}[\bar{\beta}]$ are the complete set of the representatives of the K_{l-1} -orbits in $X(\psi_\beta)$ which contains some element of $X_0(\psi_\beta)$. This means that Hypothesis 3.2.1 is valid if $\bar{\beta} \in M_n(\mathbb{F})$ is regular split and semi-simple (that is the case where the argument in the proof of Theorem 4.6 of [7] works), but this is a special case of Theorem 5.2.1.

The examples 4.6.2, 4.6.3, 4.6.4 show

Theorem 5.3.1 *Hypothesis 3.2.1 is valid if $n_i \leq 4$ ($i = 1, \dots, r$) and $\text{ch } \mathbb{F} > 7$.*

These results strongly suggest that Hypothesis 3.2.1 is valid if $\overline{\beta} \in M_n(\mathbb{F})$ is regular and the characteristic of \mathbb{F} is big enough. We will discuss on this subject further in the forthcoming paper.

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Sendai 980-0845, Japan
Miyagi University of Education
Department of Mathematics
e-mail:k-taka2@ipc.miyakyo-u.ac.jp